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Some Considerations on the Cut Locus of a Riemannian Manifold

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§ 0. Introduction

Let (M, g) be a compact connected Riemannian manifold of dimension n and fix a point p of M. Let γ_X be a geodesic emanating from p with the unit initial direction $X \in T_pM$. We define the cut point of p along γ_X as the last point on γ_X to which the geodesic minimizes the distance. The locus $C(p)$ of all cut points of p is called the cut locus of p.

By the above definition M is obtained from $C(p)$ by attaching an n-cell and the cut locus contains the essential informations on the topology of M. Now the problem of determining the structure of the cut locus is interesting in connection with the singularity theory. Recently in case of analytic Riemannian structures or in generic case much progress has been made by M. Buchner ([2] [3] [4]). But since their works appeal to the powerful general theory (Hironaka's or Mather's theory), the concrete structure of the cut locus is not given explicitly.

On the other hand the above problem is answered for the 2-dimensional analytic case by S.B. Myers ([8]), symmetric spaces and Berger's spheres by T. Sakai ([11] [12]) and M. Takeuchi ([13]). But with respect to an arbitrary metric, the cut locus may be very complicated, for example, H. Gluck, D. Singer ([5]) showed that there exists a metric on any manifold whose cut locus is not triangurable.

The main purpose of the present paper is to study the relation between the cut locus and the union of all unstable manifolds of critical points with positive index of some Morse functions.

Firstly in Section 1 we approximate the distance function from p by Morse function with respect to C^0-topology and define the set $C^1(p)$ as the limit set of all unstable manifolds of critical points with positive index of Morse functions. Then $C^1(p)$ is contained in the cut locus of p and inherits the essence of the topology of M under some conditions. We call $C^1(p)$ the essential cut locus. In general it seems that the structure of

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