Advanced Studies in Pure Mathematics 2, 1983
Galois Groups and their Representations
pp. 89–97

On Unramified Extensions of Function Fields over Finite Fields

Yasutaka Ihara

Let k be an algebraic function field of one variable with genus g over a finite constant field $\boldsymbol{F}_q$, and S be a given *non-empty* set of prime divisors of k. Denote by k_S^{ur} the maximum unramified Galois extension of k in which all prime divisors of k belonging to S decompose completely. Since S is nonempty, the algebraic closure of $\boldsymbol{F}_q$ in k_S^{ur} must be finite over $\boldsymbol{F}_q$. In this report, we shall give a survey of our results on this type of extensions k_S^{ur}.

§ **1.***) First, one expects that if k_S^{ur}/k is an *infinite* extension, then S cannot be "too big". What is the natural quantitative result along this line? The Chebotarev density of S is of course 0, but we need a stronger result. By studying the behaviour of zeta functions of intermediate fields of k_S^{ur}/k near $s=\frac{1}{2}$, using the Weil's Riemann hypothesis for curves, we obtained the following

Theorem 1. *Suppose that M is an infinite unramified Galois extension of k. For each prime divisor P of k, let* deg P *denote its degree over $\boldsymbol{F}_q$, put $N(P)=q^{\deg P}$, and let $f(P)$ $(1\leq f(P)\leq\infty)$ denote the residue extension degree of P in M/k. Let $g\geq 1$. Then*

$$(1.1)\qquad \sum_{\substack{P\\ f(P)<\infty}} \frac{\deg P}{N(P)^{\frac{1}{2}f(P)}-1}\leq g-1\,,$$

the series on the left being convergent.

Corollary 1. *If k_S^{ur}/k is infinite, then*

$$(1.2)\qquad \sum_{P\in S} \frac{\deg P}{N(P)^{1/2}-1}\leq g-1\,.$$

In particular,

Received January 6, 1983.

*) The results of § 1 are obtained after the Symposium. Details will appear in [Ih 7].