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On Generalized Hasse-Witt Invariants of an Algebraic Curve

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§ 1. Introduction

Let k be an algebraically closed field of characteristic $p>0$, and C a connected complete non-singular curve over k. Denote by $\pi_1(C)$ the Grothendieck fundamental group of C. (cf. [3] exp. V. The group $\pi_1(C)$ is isomorphic to Gal (K_{ur}/K), where K is the function field of C and K_{ur} means the maximal unramified extension field of K.) Concerning this group $\pi_1(C)$, we shall generalize the result of Katsurada [7] (Theorem 1 in Section 2) and then prove another related theorem (Theorem 2 in Section 4).

To begin with, a short account will be given on the known facts about the structure of the group $\pi_1(C)$. For a non-negative integer g, put $\Gamma_g=\langle a_1, \cdots, a_g, b_1, \cdots, b_g \,|\, a_1b_1a_1^{-1}b_1^{-1}\cdots a_gb_ga_g^{-1}b_g^{-1}=1\rangle$, the group generated by $2g$ elements $a_1, \cdots, a_g, b_1, \cdots, b_g$ with one defining relation $a_1b_1a_1^{-1}b_1^{-1}\cdots a_gb_ga_g^{-1}b_g^{-1}=1$. ($\Gamma_g=\{1\}$ if $g=0$.) The group Γ_g is nothing but the topological fundamental group of a Riemann surface of genus g. Further, let $\hat{\Gamma}_g$ be the pro-finite completion of Γ_g, i.e. $\hat{\Gamma}_g=\varprojlim(\Gamma_g/\Gamma)$ where Γ ranges over all normal subgroups of Γ_g with finite indices. Then, we can state a fundamental result of Grothendieck about $\pi_1(C)$ ([3] exp. X): If the genus of C equals g, then there exists a surjective continuous homomorphism $\varphi\colon \hat{\Gamma}_g\to\pi_1(C)$ with the following property:

(∗) Ker φ is contained in every open normal subgroup N of $\hat{\Gamma}_g$ such that $[\hat{\Gamma}_g: N]$ is prime to p.

The surjectivity of φ says that to each finite étale covering $C'\to C$ there corresponds a unique open subgroup N of $\hat{\Gamma}_g$. (The correspondence is given by $N=\varphi^{-1}(\pi_1(C'))$.) And the property (∗) ensures that each open normal subgroup N of $\hat{\Gamma}_g$ with $[\hat{\Gamma}_g: N]$ prime to p can be obtained as $\varphi^{-1}(\pi_1(C'))$ for some connected étale covering $C'\to C$. But how about the groups N for which $[\hat{\Gamma}_g: N]$ is divisible by p? Or, we naturally ask a

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