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Hodge Structures of Shimura Varieties Attached to the Unit Groups of Quaternion Algebras

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Introduction

a) Let G be a reductive algebraic group defined over an algebraic number field F. Then it is believed that for any "geometric" primitive modular form f over the adelization G_A of G, there exists a motif M_f naturally attached to f (cf. Deligne [1]). It is already one question to give a precise meaning to the term "geometric". But one might set aside this temporarily, by agreeing to the following. When F is totally real and the quotient space $(G\times_{F,\sigma}\boldsymbol{R})(\boldsymbol{R})/K$ of the real Lie group of the real points of $G\times_{F,\sigma}\boldsymbol{R}$ by a maximal compact group K of $(G\times_{F,\sigma}\boldsymbol{R})(\boldsymbol{R})$, has an invariant complex structure for any embedding: $\sigma\colon F\hookrightarrow \boldsymbol{R}$, those modular forms f which belong to the automorphic representations $\pi=\otimes_v \pi_v$ of G_A with infinite components π_σ belonging to the discrete series of $(G\times_{F,\sigma}\boldsymbol{R})(\boldsymbol{R})$, are geometric.

Then it is a problem to know how to attach motives M_f to the primitive forms f of the above type. We want to discuss this problem for modular forms f of weight 2 with respect to the Hilbert modular groups or the unit groups of quaternion algebras, i.e. when F is totally real and $G=GL(2)$ or $G=B^\times$ for some indefinite quaternion algebra B over F. This case is very special. Nevertheless, the answer is by no means trivial as we shall see soon. In this paper, we announce some results on the Hodge structures attached to f, which are generalizations of the results of the previous paper [9].

b) In order to explain the purpose of this paper, we start with some conjectures proposed in [9], which we reformulate here, employing the terminology of motives in Deligne [2].

Let F be a totally real algebraic number field of degree g over $\boldsymbol{Q}$. Let us call a modular cusp form f over the adelization $GL_2(A)$ of $GL_2(F)$ a primitive Hilbert modular cusp form, if it generates an irreducible automorphic representation $\pi_f=\otimes_v \pi_v$ of $GL_2(A)$ whose infinite component π_σ belongs to the discrete series of $GL_2(\boldsymbol{R})$ for any embedding $\sigma\colon F\hookrightarrow\boldsymbol{R}$,

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