Advanced Studies in Pure Mathematics 1, 1983
Algebraic Varieties and Analytic Varieties
pp. 355–384

Standard Monomial Theory and the Work of Demazure

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In collaboration with V. Lakshmibai and C. Musili (cf. [8], [9], [10], [12]), we have given a generalization of the classical Hodge-Young standard monomial theory (cf [4], [5]) of $SL(n)$ to the case of an arbitrary semi-simple linear algebraic group G. The purpose of this generalization is to give an explicit basis for the space $H^0(G/B, L)$ of sections of a line bundle L (associated to a dominant weight) on the flag variety G/B, or more generally for $H^0(X, L)$, where X is a Schubert variety in G/B. Our results provide a complete solution to this problem, when G is classical and only partial answers when G is exceptional. Recall that when the base field is of characteristic zero (Borel-Weil theorem) every irreducible G-module is of the form $H^0(G/B, L)$ (L as above), so that a particular case of this problem is to give an explicit basis for an irreducible G-module. A survey of our results, including its motivation and applications, has been given in [11].

Our first purpose here is to give a proof of the first theorem on standard monomial theory, namely the basis theorem for a fundamental representation (say the field is of characteristic zero), such that its highest weight ω is of *classical type* (see Theorem 3). The proof of this theorem, given here, is not really different from the one given in G/P-IV (cf [9]). However, we have separated out many general considerations with which it is mixed up in [9] and this may be of help in understanding this theorem.

The work of Demazure (cf [2]) is basic to the proof of the main results of standard monomial theory (cf. G/P-IV, [9]), especially his character formula which generalizes the Weyl character formula. Further, standard monomial theory can be considered as a refined version of a conjecture made by Demazure in [2] (see also Remarks 4, 5 and 6). Our second purpose here is to give a fairly self-contained exposition of the results of Demazure, relevant to standard monomial theory. Our exposition of this work of Demazure (see § 2) is basically the same as his; however, it avoids his big inductive machinery (which is perhaps necessary for the desingularisation of Schubert varieties). Consequently, the proofs given here of his vanishing theorems and character formula for line bundles on Schubert

Received December 20, 1981.