CHAPTER 3

Uniform Tightness and Asymptotic Tightness

1. Introduction

Any limit theory deals with the notion of compactness through the existence or not for sequences of sub-sequences converging in the sense of the defined limit. This corresponds to the Bolzano-Weierstrass for real sequences. For the weak convergence, the condition of the existence of such sub-sequences is called tightness . When dealing with weak convergence for general metric spaces, tightness leads to the general Prohorov theorem which establishes, under eventually other assumptions, that every uniformly tight sequence of measurable applications of a metric space (S, d) has at least a weakly converging sub-sequence.

In this chapter, we focus on weak convergence in $\mathbb{R}^k$. And there exists a specific handling of weak compactness that is very different from the treatment in the general case. In $\mathbb{R}^k$, the major role is played by the theorem of Helly-Bray that directly makes use of the Bolzano-Weierstrass theorem in $\mathbb{R}$.

Since, we deal with compact sets of $\mathbb{R}^k$, just remind two properties which we are going to use. The first is that compact sets of $\mathbb{R}^k$ are closed and bounded sets . The second is that $\mathbb{R}^k$ is a complete and separable metric space .

Here, we will be mainly dealing with the max-norm defined for $x = (x_1, \cdots, x_k) \in \mathbb{R}^k$ by

$$\|x\| = \max_{1 \leq i \leq k} |x_i|.$$

The open balls $B(x, r)$ and the closed balls $B^f(x, r)$ with respect to this norm are