CHAPTER 2

# Weak Convergence Theory

## 1. Introduction

In this chapter, we treat a unified theory of weak convergence by its functional characterization. We want to have complete theory of limits of sequences of probability measures on $(\mathbb{R}^k, \mathcal{B}(\mathbb{R}^k))$, where $\mathcal{B}(\mathbb{R}^k)$ is the Borel $\sigma$-algebra on $\mathbb{R}^k$.

However, the handling of the fundamental results only uses the metric structure of $\mathbb{R}^k$. This is why, whenever possible, we deal with sequences of probability measures on a metric spaces $(S, d)$, endowed with its Borel $\sigma$-algebra $\mathcal{B}(S)$.

But when dealing with limits of sub-sequences of sequences of random variables or probability measures, we essentially remain in $\mathbb{R}^k$ by making profit of the Helly-Bray theorem .

As in any theory on limits, we will have to deal with the uniqueness of limits, and convergence criteria, and relative compactness. Here, we will speak of weak compactness or simply tightness or uniform tightness.

## 2. Definition, Uniqueness and Portmanteau Theorem

DEFINITION 1. *The sequence of measurable applications* $X_n : (\Omega_n, \mathcal{A}_n, \mathbb{P}_n) \mapsto (S, B(S))$ *weakly converges to the measurable application* $X : (\Omega_\infty, \mathcal{A}_\infty, \mathbb{P}_\infty) \mapsto (S, \mathcal{B}(S))$ *if and only for any continuous and bounded function* $f : S \mapsto \mathbb{R}$*, (denoted* $f \in \mathcal{C}_b(S)$*), we have*

$$\mathbb{E}f(X_n) \to \mathbb{E}f(X) \text{ as } n \to +\infty. \tag{2.1}$$

We notice that the spaces on which the applications $X_n$ are defined have no importance here. Only matter their probability laws on $(S, d)$. Indeed, denote $L = \mathbb{P}_X = \mathbb{P}_\infty \circ X^{-1}$, the probability law of $X$ defined by