*Multivariate Analysis and Its Applications*
IMS Lecture Notes – Monograph Series (1994) Volume 24

# THE CIRCULAR FUNCTIONAL MODEL

BY N. N. CHAN AND TAK K. MAK
*Chinese University of Hong Kong and Concordia University*

Consistent estimation of the structural parameters in a circular functional relationship between two variables observed with error is given, and a practical procedure for computing estimates is proposed.

**1. Introduction.** The circular relationship model is defined as follows. Let $x_i = \xi + \rho\cos\tau_i + \delta_i$, $y_i = \eta + \rho\sin\tau_i + \varepsilon_i$ $(i = 1, \cdots, n)$, where the $n$ pairs $(\delta_1, \varepsilon_1), \cdots, (\delta_n, \varepsilon_n)$ are independent and normally distributed with zero mean and covariance matrix $\sigma^2 I$ ($I$ being the identity matrix of order 2), $\xi, \eta, \rho$ and $\sigma^2$ are the structural parameters, and the $\tau_i$ are either incidental parameters in the functional model or independent and identically distributed in the structural model. Chan (1965) used the method of minimum squared distance to estimate the structural parameters in the functional model and considered consistency of these estimates. Anderson (1981) examined the maximum likelihood estimation of the parameters, assuming a uniform distribution over [0, $2\pi$) for the $\tau_i$, and highlighted the difficulties involved in an adequate analysis of the functional model. Berman (1983) estimated the parameters of a circle when angular differences between successive data points are known. Both of them considered the data set of the Brogar ring given by Thom and Thom (1973). In §2, we modify the estimating equations of Chan (1965, equations (8)) to yield a set of consistent estimators of the structural parameters without invoking the additional assumptions of a uniform distribution for the $\tau_i$ or of known angular differences, and indicate the way to derive their asymptotic covariance matrix. Our method of estimation can also be applied to data that lie on a small section of an arc, as, e.g., given in the survey data of the stone 'circles' at Avebury by Thom, Thom and Foord (1976). A practical procedure for computing these estimates is provided in §3 and illustrated by the numerical examples given in Chan (1965, §6) and a Monte Carlo simulation, and by the above two data sets in archaeology.

---

AMS 1980 Subject Classification:Primary 62J99; Secondary 62F.

Key words and phrases: Circular functional relationship, circular structural model, confluent hypergeometric function, minimum distance estimation.