ISING MODELS AND DEPENDENT PERCOLATION

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An intimate relation between Ising and certain dependent percolation models was discovered some twenty years ago by Kasteleyn and Fortuin and developed more recently by Swendsen and Wang. We review this relation and the role of stochastic domination within it. When the Ising model is not ferromagnetic (i.e., not positively dependent), the related percolation model is more complicated but still of interest.

1. Introduction. Although percolation and more recently Ising models have been of interest to probabilists and statisticians for some time, they have been largely unaware of the beautiful and useful relation which exists between the two types of systems. This relation, originally discovered by Kasteleyn and Fortuin (1969) was clarified by recent work of Swendsen and Wang (1987) on Ising simulation methods, and further explained by Edwards and Sokal (1988). Our purpose here is to describe the relation (Section 2), explain how it yields certain stochastic monotonicity properties (Section 3) of Fortuin (1972) and then mention some applications due to Aizenman, Chayes, Chayes and Newman (1987, 1988). We also discuss (Section 5) the situation when one goes beyond the case of ferromagnetic (i.e., positively dependent) Ising models. It should be noted that most (all?) of what is presented in this paper satisfies one or more of the following descriptions: old, already published, known to the experts. For more details related to Sections 3 and 4 (and much of Section 2), see Aizenman, Chayes, Chayes and Newman (1988).

2. Random Colored Graphs – Positively Dependent Case. Let Λ be a finite set of sites (or vertices) and $\mathcal{B}$ the corresponding set of all bonds (or edges); i.e., $\mathcal{B}$ is the set of pairs $b = \{x, y\}$ of sites. (For many applications Λ is a subset of some regular d-dimensional lattice, say $\mathbf{Z}^d$, and one takes $\Lambda \uparrow \mathbf{Z}^d$.) We will consider bond random variables n_b taking values 0 or 1 and site random variables T_x taking values in $\{1, \cdots, q\}$. The n_b's define a random graph with vertex set Λ in which

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