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Derivability of Some Operations on Distribution Functions

By Roger B. Nelsen, José Juan Quesada-Molina,[1] Berthold Schweizer,[2] and Carlo Sempi[1]
Lewis and Clark College, Universidad de Granada, University of Massachusetts, and Università di Lecce

In this paper we characterize the operations on distribution functions that are both derivable from functions on random variables defined on a common probability space and induced pointwise by functions from $[0,1]^n$ into $[0,1]$. We specify the class of functions on random variables from which the operations are derived and show that it includes all order statistics; and we give a description of the n-place functions from which these operations are induced pointwise. In addition, by way of illustration, we show that mixtures, which are induced pointwise, are not derivable.

1. Preliminary Concepts and Results. We shall denote by $\mathcal{D}$ the space of proper one-dimensional distribution functions (d.f.'s), i.e. the space of functions $F : \overline{\mathbf{R}} := [-\infty, +\infty] \to [0,1]$ that are nondecreasing, left-continuous on $\mathbf{R} := (-\infty, +\infty)$ and such that

$$F(-\infty) = 0 = \lim_{x \to -\infty} F(x) \text{ and } F(+\infty) = 1 = \lim_{x \to +\infty} F(x).$$

An n-operation ϕ on $\mathcal{D}$ is a mapping from $\mathcal{D}^n := \mathcal{D} \times \mathcal{D} \times \cdots \times \mathcal{D}$ into $\mathcal{D}$, i.e., a mapping that assigns a d.f. to every ordered collection of n d.f.'s. If X_i is a random variable (r.v.), we shall denote the distribution function of X_i by F_i, F_{X_i}, or $df(X_i)$, whichever is more convenient.

Definition 1.1. An n-operation ϕ on $\mathcal{D}$ is said to be *derivable* from a function on r.v.'s if there exists a Borel measurable function V from $\overline{\mathbf{R}}^n$ into $\overline{\mathbf{R}}$ that

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