SECTION 2

Symmetrization and Conditioning

In this section we begin the task of bounding $\mathbb{P}\,\Phi(\sup_t |S_n(\cdot,t) - M_n(t)|)$ for a general convex, increasing function Φ on $\mathbb{R}^+$. The idea is to introduce more randomness into the problem and then work conditionally on the particular realization of the $\{f_i\}$. This is somewhat akin to the use of randomization in experimental design, where one artificially creates an extra source of randomness to ensure that test statistics have desirable behavior conditional on the experimental data.

As a convenience for describing the various sources of randomness, suppose that the underlying probability space $(\Omega, \mathcal{A}, \mathbb{P})$ is a product space,

$$\Omega = \Omega_1 \otimes \cdots \otimes \Omega_n \otimes \Omega_1' \otimes \cdots \otimes \Omega_n' \otimes \mathcal{S},$$

equipped with a product measure

$$\mathbb{P} = \mathbb{P}_1 \otimes \cdots \otimes \mathbb{P}_n \otimes \mathbb{P}_1' \otimes \cdots \otimes \mathbb{P}_n' \otimes \mathbb{P}_\sigma.$$

Here $\Omega_i' = \Omega_i$ and $\mathbb{P}_i' = \mathbb{P}_i$. The set $\mathcal{S}$ consists of all n-tuples $\boldsymbol{\sigma} = (\sigma_1, \ldots, \sigma_n)$ with each σ_i either $+1$ or -1, and $\mathbb{P}_\sigma$ is the uniform distribution, which puts mass 2^{-n} on each n-tuple.

Let the process $f_i(\cdot,t)$ depend only on the coordinate ω_i in Ω_i; with a slight abuse of notation write $f_i(\omega_i,t)$. The Ω_i' and $\mathbb{P}_i'$ are included in order to generate an independent copy $f_i(\omega_i',t)$ of the process. Under $\mathbb{P}_\sigma$, the σ_i are independent sign variables. They provide the randomization for the symmetrized process

$$S_n^\circ(\omega,t) = \sum_{i\le n} \sigma_i f_i(\omega_i,t).$$

We will find that this process is more variable than S_n, in the sense that

$$\mathbb{P}\,\Phi(\sup_t |S_n(\cdot,t) - M_n(t)|) \le \mathbb{P}\,\Phi(2\sup_t |S_n^\circ(\cdot,t)|) \tag{2.1}$$

The proof will involve little more than an application of Jensen's inequality.