CHAPTER 13

Density Ratio of a Maximal Invariant

13.1. Density ratio as a ratio of integrals over the group. There are statistical problems, especially testing problems, where the object of interest is the density ratio of two distributions of a maximal invariant $T = t(X)$, say P_1^T and P_2^T, rather than the individual distributions P_i^T themselves. If Theorem 8.6 applies, and if for $i = 1, 2$, P_i is a distribution on $\mathfrak{X}$ with density p_i with respect to a χ-relatively invariant measure λ as in (8.11), then (8.12) shows that

$$\text{(13.1.1)} \qquad \frac{dP_2^T}{dP_1^T}(t) = \frac{\int p_2(gs(t))\chi(g)\mu_G(dg)}{\int p_1(gs(t))\chi(g)\mu_G(dg)}.$$

It is seen that the measure $\mu_{\mathfrak{T}}$ in (8.12) drops out, so that it is unnecessary to deal with the factorization (8.10). We can go a step further. In (13.1.1) replace g by gg_1 with any fixed $g_1 \in G$, and observe $\chi(gg_1) = \chi(g)\chi(g_1)$ and $\mu_G(dgg_1) = \Delta_r(g_1)\mu_G(dg)$, in which Δ_r is the right-hand modulus of G (Section 7.1). Since $\chi(g_1)\Delta_r(g_1)$ can be taken outside the integrals in (13.1.1), it drops out of the ratio. The result is that on the right-hand side of (13.1.1) $s(t)$ is replaced by $g_1 s(t)$. Since $g_1 \in G$ is arbitrary, $g_1 s(t)$ is an arbitrary point on the G-orbit of $s(t)$. Replacing this point by x, (13.1.1) can be written

$$\text{(13.1.2)} \qquad \frac{dP_2^T}{dP_1^T}(t(x)) = \frac{\int p_2(gx)\chi(g)\mu_G(dg)}{\int p_1(gx)\chi(g)\mu_G(dg)}$$

for any $x \in \mathfrak{X}$. Thus, it is not necessary to construct a function s with range a global cross section $\mathfrak{Z}$. The right-hand side of (13.1.2)