CHAPTER 5

Lie Groups and Lie Algebras

5.1. Definition of a Lie group and examples. A **Lie group** is a group G that is at the same time an analytic manifold such that the group multiplication $G \times G \to G$ defined by $(g_1, g_2) \to g_1 g_2$ is analytic. It can be shown that then the mapping $g \to g^{-1}$ of $G \to G$ is also analytic (Cohn, 1957, Theorem 2.6.1). Equivalently, the multiplication $(g_1, g_2) \to g_1 g_2^{-1}$ can be required to be analytic. (Some authors replace "analytic" by "C^∞" in the above definition.)

Left and right translation. For fixed $g \in G$, the transformation $h \to gh$, $h \in G$, is a 1-1 transformation of G onto itself which is an analytic diffeomorphism. It is called a **left translation** with g and denoted L_g. Similarly, **right translation** with g, denoted R_g, is the transformation $h \to hg$, $h \in G$. Any chart at the identity element e of G can serve also as a chart at an arbitrary element $g \in G$ by transporting it with help of the left translation L_g (or by the right translation R_g). It is therefore usually sufficient to consider only charts at e.

An obvious example of a Lie group is R^d with group multiplication defined as vector addition. Here the whole group can be covered by one chart (the usual coordinate system) and analyticity of the group multiplication is immediate. Other examples that are very important for statistical applications are provided by matrix groups, i.e., the **general linear group** $GL(n)$ of all $n \times n$ real nonsingular matrices, and its Lie subgroups. Among the latter especially important are **orthogonal**, **triangular**, **diagonal**, and **block diagonal** matrices.