CHAPTER 4

Differential Forms on Manifolds

4.1. Grassmann algebra. It may be helpful to precede the formal definition by a short informal discussion. We shall keep a given differentiable manifold M and an arbitrary point $p \in M$ fixed throughout this section. Let f be a C^1 function $M \to R$ and df its differential. It was seen in Chapter 3 that one of the two possible interpretations of the value of df at p is a linear functional on the tangent space M_p. That is, df at p is a member of the space M_p^* dual to M_p. In differential geometry there is a need for functions whose arguments consist of more than one element of M_p. The case of two elements is especially prevalent, for instance in the notions of curvature and torsion transformations (see Bishop and Crittenden, 1964). In this monograph the most important case will be d arguments, where $d = \dim M$, since that will be used to construct a measure on M (Section 6.6). In general, then, we are going to define, for every $1 \leq k \leq d$, a differential form ω of degree k, or, simply, a k-form. Its value at p is denoted ω_p and will be defined as a real valued function of a certain kind on the k-fold product $M_p \times \cdots \times M_p$. (The extension to ω_p being vector valued is important in differential geometry, but not for the purpose of this monograph.)

In order to simplify the notation put $W^k = M_p^{\times k} = k$-fold product of M_p with itself ($k \geq 1$). A real valued function u on W^k is said to be **k-linear** if u is linear in each of its k arguments separately. The function u is called **alternating** if u changes sign whenever two arguments are interchanged. Equivalently, this can be expressed in