4. GENERAL SADDLEPOINT APPROXIMATIONS

4.1. INTRODUCTION

In this chapter, we use the approximations obtained for the mean to get approximations for more complicated statistics. The first section considers one-dimensional M-estimates i.e. T_n is the solution of $\sum_1^n \psi(x_i;t) = 0$. The estimate T_n is written locally as a mean and the saddlepoint approximation for the mean is used. In section 4.3, we consider a slightly different approach in that the moment generating function is approximated and a saddlepoint approximation used. The technique is applied to approximating the density of L-estimates in the next section. At this point, we turn to the problem for multivariate M-estimates. Techniques are similar to those used for one-dimensional M-estimates. Finally we modify the results to handle the case of regression using M-estimates. Throughout the chapter, there are numerical results illustrating the accuracy of these approximations even for small sample sizes.

In the cases considered in this chapter, our interest is to be able to say something about the density of an estimate. Although asymptotic results are available in most cases, we usually do not know whether these asymptotic distributions are good approximations for small or moderate sample sizes. For instance there are several proposals for using "t-statistics" based on robust location/scale estimates as a means for computing confidence intervals. We then need to know whether this t-approximation works reasonably and if it does, what are appropriate degrees of freedom. Some results in this direction are given in section 4.5.b.

4.2. ONE-DIMENSIONAL M-ESTIMATORS

To begin, consider the problem of finding a saddlepoint approximation for the density of a one-dimensional M-estimate. As developed by Huber (1964, 1967) M-estimates are defined as the solution T_n of

$$\sum_{i=1}^{n} \psi(x_i, t) = 0 \tag{4.1}$$

for observations $x_1, x_2, \cdots, x_n$. If the x_i's are independent observations from a density $f(x,\theta)$, then by setting $\psi(x,\theta) = \frac{\partial}{\partial\theta} \log f(x,\theta)$, T_n becomes the maximum likelihood estimate of θ. In much of the work on robustness, M-estimates play a central role. However the derivation of the exact density of such an estimate is usually intractable mathematically and it becomes essential to have a good approximation in order to carry out inference.

Denote the density of T_n when the x_i's are independent observations from a density f as $f_n(t)$. To approximate $f_n(t)$, we proceed by writing T_n as a mean up to a certain order and then using the saddlepoint approximation to the mean as derived in section 3.2. The approach follows closely that developed in Field (1982) for multivariate M-estimates which in turn uses critically results on multivariate Edgeworth expansions in Bhattacharya and Ghosh (1978). Field and Hampel (1982) give an alternate derivation for univariate M-estimates based on the log-derivative density. In this section, the approximation is developed for a one-dimensional M-estimate. The development for multivariate M-estimates is presented in section 4.5 .

In the development of the approximation, the conjugate density (cf. (3.22)) will play