3. SADDLEPOINT APPROXIMATIONS FOR THE MEAN

3.1. INTRODUCTION

The goal of this chapter is to introduce saddlepoint techniques for a simple problem, namely the approximation to the distribution of the mean of n iid random variables.

Although this case does not have much relevance from a practical point of view, the same basic idea is used in more complex models to derive saddlepoint approximations for very general statistics; cf. chapters 4 and 5. Thus, a good understanding of the technique in this simple case will allow a direct application to more complex and important situations. Historically, this was the first explicit statistical application of this method. It was developed by H.E. Daniels in a fundamental paper in 1954.

Basically, there are two ways to derive a saddlepoint approximation. The first one is presented in section 3.3 and is an application of the method of steepest descent (section 3.2). The second one is based on the idea of recentering by means of a conjugate or associate distribution (section 3.4) and shows the connection between saddlepoint techniques and Edgeworth expansions. Both ways lead to the same approximation and from a methodological point of view they both have their own merits. Finally, the examples in section 3.5 show the great accuracy of these approximations for very small sample sizes and far out in the tails.

3.2. THE METHOD OF STEEPEST DESCENT

We discuss here a general technique which allows us to compute asymptotic expansions of integrals of the form

$$\int_{\mathcal{P}} e^{v \cdot w(z)} \xi(z) dz \tag{3.1}$$

when the real parameter v is large and positive. Here w and ξ are analytic functions of z in a domain of the complex plane which contains the path of integration $\mathcal{P}$. This technique is called the *method of steepest descent* and will be used to derive saddlepoint approximations to the density of a mean (section 3.3) and later of a general statistic (chapter 4). In our exposition we follow Copson (1965). Other basic references are DeBruijn (1970), and Barndorff-Neilsen and Cox (1989).

Consider first the integral (3.1). In order to compute it we can deform arbitrarily the path of integration $\mathcal{P}$ provided we remain in the domain where w and ξ are analytic. We deform $\mathcal{P}$ such that

(i) the new path of integration passes through a zero of the derivative $w'(z)$ of w;

(ii) the imaginary part of w, $\Im w(z)$ is constant on the new path.

Let us now look at the implications of (i) and (ii). If we write

$$z = x + iy, \qquad z_0 = x_0 + iy_0,$$

$$w(z) = u(x, y) + iv(x, y), \qquad w'(z_o) = 0,$$

and denote by S the surface $(x, y) \mapsto u(x, y)$, then by the Cauchy-Riemann differential equations

$$u_x = v_y, \qquad u_y = -v_x,$$