2. EDGEWORTH EXPANSIONS

2.1. INTRODUCTION

Given a central limit theorem result for a statistic, one may hope to obtain an estimate for the error involved, that is the difference between the exact distribution $F_n(t)$ of the standardized statistic and the standard normal distribution $\Phi(t)$. Typically, this result for the distribution of the sum of n iid random variables is known as the Berry-Esseen theorem and takes the form

$$\sup_t |F_n(t) - \Phi(t)| \leq \frac{C_0}{\sqrt{n}}, \tag{2.1}$$

where the constant C_0 depends on the statistic and on the underlying distribution of the observations but not on the sample size n. We will discuss this result for the simplest case and mention some generalizations in section 2.2.

The inequality (2.1) suggests a way to improve the approximation of F_n by considering a complete *asymptotic expansion* of the form

$$\sum_{j=0}^{\infty} \frac{A_j(t)}{n^{j/2}}, \tag{2.2}$$

where the error incurred by using the partial sum is of the same order of magnitude as the first neglected term,
i.e.

$$|F_n(t) - \sum_{j=0}^{r} \frac{A_j(t)}{n^{j/2}}| \leq \frac{C_r(t)}{n^{(r+1)/2}}. \tag{2.3}$$

Of course, in our case $A_0(t) = \Phi(t)$ the cumulative of the standard normal distribution and $C_0(t) \equiv C_0$ is the constant given by the Berry-Esseen theorem. Note that for any fixed n and t (2.2) may or may not exist and that we are just using the property (2.3) of the partial sums to approximate F_n. These asymptotic expansions are common in numerical analysis where they are used to approximate a variety of special functions, including Bessel functions. For a good discussion of theoretical and numerical aspects, see Henrici (1977), Ch. 11. The following example presents some typical numerical aspects of these approximations.

Example 2.1

We consider the approximation of the Binet function

$$J(z) = \log\Gamma(z) - \frac{1}{2}\log(2\pi) - (z - \frac{1}{2})\log z + z$$

through the partial sums of the asymptotic expansion

$$\sum_{j=1}^{\infty} \frac{B_{2j}}{2j(2j-1)} z^{-2j+1},$$

where $\Gamma(z)$ is the Gamma function and B_{2j} are the Bernoulli numbers given by

2j	2	4	6	8	10	12	...
B_{2j}	1/6	-1/30	1/42	-1/30	5/66	-691/2730	...