1. INTRODUCTION

1.1. MOTIVATION

Suppose we are interested in the density f_n of some statistic T_n $(x_1, \cdots, x_n)$, where $x_1, \cdots, x_n$ are n independent identically distributed (iid) observations with the underlying density f . Unless T_n and/or f have special forms, one cannot usually compute analytically the distribution of T_n.

A first alternative is to rely on asymptotic theory. Very often one can "linearize" the statistic T_n and prove that the linearized statistic is equivalent to T_n as $n \to \infty$, that is the difference goes to zero in probability. This leads through the central limit theorem to many asymptotic normality proofs and the resulting asymptotic distribution can be used as an approximation to the exact distribution of T_n. This is certainly a powerful tool from a theoretical point of view as can be seen in some good books on the subject, e.g. Bhattacharya and Rao (1976), Serfling (1980); cf. also the innovative article by Pollard (1985). But, in spite of the fact that in some complex situations one does not have any viable alternatives, very often the asymptotic distribution does not provide a good approximation unless the sample size is (very) large. Moreover, these approximations tend to be inaccurate in the tails of the distribution.

Many techniques have been devised to increase the accuracy of the approximation of the exact density f_n. A well known method is to use the first few terms of an *Edgeworth expansion* (cf. for instance Feller, 1971, Chapter 16). This is an expansion in powers of $n^{-1/2}$, where the leading term is the normal density. It turns out in general that the Edgeworth expansion provides a good approximation in the center of the density, but can be inaccurate in the tails where it can even become negative. Thus, the Edgeworth expansion can be unreliable for calculating tail probabilities (the values usually of interest) when the sample size is moderate to small.

In a pioneering paper, H.E. Daniels in 1954 introduced a new type of idea into statistics by applying *saddlepoint techniques* to derive a very accurate approximation to the distribution of the arithmetic mean of $x_1, \cdots, x_n$. The key idea is as follows. The density f_n can be written as an integral on the complex plane by means of a Fourier transform. Since the integrand is of the form $\exp(nw(z))$, the major contribution to this integral for large n will come from a neighborhood of the saddlepoint z_0, a zero of $w'(z)$. By means of the *method of steepest descent*, one can then derive a complete expansion for f_n with terms in powers of n^{-1}. Daniels (1954) also showed that this expansion is the same as that obtained using the idea of the *conjugate density* (see Esscher, 1932; Cramér, 1938; Khinchin, 1949) which can be summarized as follows. First, recenter the original underlying distribution f at the point t where f_n is to be approximated; that is, define the conjugate (or associate) density of f, h_t. Then use the Edgeworth expansion locally at t with respect to h_t and transform the results back in terms of the original density f. Since t is the mean of the conjugate density h_t, the Edgeworth expansion at t with respect to h_t is in fact an expansion in powers of n^{-1} and provides a good approximation locally at that point. Roughly speaking, a higher order approximation around the center of the distribution is replaced by local low order approximations around each point. The unusual characteristic of these expansions is that the first few terms (or even just the leading term) often give very accurate approximations in the far tails of the distribution even for very small sample sizes. Besides the theoretical reasons, one empirical reason for the excellent small sample behaviour is that saddlepoint approximations are density-like objects and do not show the polynomial-like waves exhibited for instance by Edgeworth approximations.