V. SUPREMA DISTRIBUTIONS

1. Introduction.

We now turn our attention to the distribution of the supremum of a centered, Gaussian X over T, which, as always, we write either as $\|X\|_T$ or simply as $\|X\|$ when there is no danger of confusion. In particular, we shall be interested in the asymptotic behavior of $P\{\|X\| > \lambda\}$ as $\lambda \to \infty$.

As we have already seen, an upper bound to this probability comes from Borell's inequality, which tells us that for all $\lambda > 0$

$$(5.1) \qquad P\{|\,\|X\| - E\|X\|\,| > \lambda\} \;<\; 2e^{\frac{1}{2}\lambda^2/\sigma_T^2},$$

where $\sigma_T^2 = \sup_T EX_t^2$. This implies that for all $\lambda > E\|X\|$

$$(5.2) \qquad P\{\|X\| > \lambda\} \;<\; 2e^{-\frac{1}{2}(\lambda^2 - 2\lambda E\|X\| + \|X\|^2)/\sigma_T^2}.$$

Since Borell's inequality was the key to finding sufficient conditions for Gaussian sample path continuity, that also turned out to be necessary, it seems reasonable to assume that it is close to sharp. Nevertheless, the aim of this chapter is to improve on (5.2) in two directions. Firstly, if we think of the right hand side of (5.2) as $f(\lambda)e^{-\frac{1}{2}\lambda^2/\sigma_T^2}$, then our aim will be to replace this by $\hat{f}(\lambda)e^{-\frac{1}{2}\lambda^2/\sigma_T^2}$, where $\hat{f}(\lambda)$ has a lower order of growth in λ than does the exponentially growing f. In many cases, it is possible to find a polynomial $\hat{f}$.

Having found such $\hat{f}$, we would also like to know if we have found the best possible, and so we shall also be interested in lower bounds for $P\{\|X\| > \lambda\}$. These almost always involve much more work than the upper bounds, so we shall generally suffice with statements without proofs. The situation is highly analagous to the continuity problem: sufficiency was easy, necessity was hard. What is rather interesting in the upper bound proofs, however, is that they proceed via a kind of "leap-froging", in which Borell's inequality is used to improve on itself!

Before we look at the general situation, however, it is worthwhile to look at an optimal situation, in which "almost everything" is known, so as to give us an idea as to what sort of results we can hope for in general.

5.1 THEOREM. *Let X be a centered, stationary Gaussian process on $\Re$, with covariance function R satisfying*

$$(5.3) \qquad R(t) \;=\; 1 \;-\; C|t|^\alpha \;+\; o(|t|^\alpha), \qquad \text{as} \quad t \to 0,$$

where $\alpha \in (0,2]$ and C are positive constants. Then for each fixed $h > 0$ such that $\sup_{\epsilon \le t \le h} R(t) < 1$ for all $\epsilon > 0$,

$$(5.4) \qquad \lim_{\lambda \to \infty} \frac{P\{\|X\|_{[0,h]} > \lambda\}}{\lambda^{2/\alpha}\Psi(\lambda)} \;=\; hC^{1/\alpha}H_\alpha,$$