IV. BOUNDEDNESS AND CONTINUITY

1. Majorising Measures.

As usual, T is the parameter space of a centered Gaussian process X, equipped with and totally bounded in the canonical metric d. Let m be a probability measure on the Borel subsets of T, and, since we shall need it very often in the following, let $g\colon (0,1] \to \Re_+$ be the function defined by

$$g(t) = \left(\log(1/t)\right)^{\frac{1}{2}}, \qquad 0 < t \leq 1. \tag{4.1}$$

Let $B(t,\epsilon)$ be an ϵ-ball in the d-metric about the point $t \in T$.

DEFINITION. *A probability measure m is called a majorising measure (for (T,d)) if*

$$\sup_{t\in T} \int_0^\infty g\big(m(B(t,\epsilon))\big)\, d\epsilon \;<\; \infty. \tag{4.2}$$

Majorising measures were formally introduced by Fernique (1974) (under the name "mesures majorantes"), although in essence they date back to a real variable lemma of Garsia, Rodemich and Rumsey (1970) and a paper of Preston (1972). Fernique proved (4.3) below, thus establishing that the existence of a majorising measure implied the boundedness of $\|X\| = \sup_{t\in T} X_t$. He continued his study of majorising measures in a number of papers, extending the ideas and coming very close to proving that they were the right tool to provide both necessary *and* sufficient conditions for continuity. For example, in Fernique (1978), he obtained both parts of Theorem 4.1 below under additional structural requirements on the space (T,d). In a pathbreaking paper, Talagrand (1987) completed Fernique's program by proving the second part of the following result in the most general case.

In this result, as with all others in this chapter, we shall assume without further comment that T has strictly positive diameter D. (Otherwise $m(B(t,\epsilon)) = 1$ for all ϵ, so that $g(m(B(t,\epsilon))) \equiv 0$, and the following result is trivial.)

4.1 THEOREM. *If m is any probability measure on (T,d), then*

$$E\|X\| \;\leq\; K \sup_t \int_0^\infty g\big(m(B(t,\epsilon))\big)\, d\epsilon, \tag{4.3}$$

for some universal constant $K \in (1,\infty)$. Furthermore, K can be chosen such that if X is bounded with probability one, then there exists a probability measure m on (T,d) satisfying

$$K^{-1} \sup_t \int_0^\infty g\big(m(B(t,\epsilon)\big)\, d\epsilon \;\leq\; E\|X\|. \tag{4.4}$$