III. PRELUDE TO CONTINUITY

1. Boundedness and Continuity.

The principle aim of this section is to show that for Gaussian processes the question of sample path continuity is intimately related to the boundedness of the supremum. In one direction this is obvious and, indeed, non-probabilistic. If the parameter space T is compact, then the a.s. continuity of X implies the a.s. boundedness of $\sup_{t\in T}|X_t|$. Thus the problem is essentially to find conditions under which processes with bounded suprema are also continuous.

Recall that we treat only centered processes, and measure continuity in terms of the canonical metric $d(s,t) = (E(X_s - X_t)^2)^{1/2}$ on T.

The first result we need is the following easy lemma, which tells us that as far as a.s. boundedness is concerned it is irrelevant whether we work with $\sup X_t$ or $\sup|X_t|$.

3.1 LEMMA. *For X centered, Gaussian, on T, and $t_o \in T$*

$$E\sup_{t\in T} X_t \;\le\; E\sup_{t\in T}|X_t| \;\le\; E|X_{t_o}| + 2E\sup_{t\in T} X_t.$$

PROOF: Only the rightmost inequality needs proving. Note the trivial inequalities that for any $t, t_o \in T$

$$X_t - X_{t_o} \;<\; \sup_{t\in T}(X_t - X_{t_o}), \qquad X_{t_o} - X_t \;<\; -\inf_{t\in T}(X_t - X_{t_o}).$$

Furthermore, both $\sup_{t\in T}(X_t - X_{t_o}) \ge 0$ and $-\inf_{t\in T}(X_t - X_{t_o}) \ge 0$. Applying the relationships $\max(a,-a) = |a|$ and $\max(a,b) \le a+b$ if $a,b \ge 0$, it follows from the above two inequalities that

$$\begin{aligned} |X_t| &\le |X_{t_o}| + |X_t - X_{t_o}| \\ &\le |X_{t_o}| + \sup_{t\in T}(X_t - X_{t_o}) - \inf_{t\in T}(X_t - X_{t_o}). \end{aligned}$$

Taking a supremum over the left hand side leaves the right side unchanged. Now take expectations, and note that symmetry gives us that $E\sup_T X_t = -E\inf_T X_t$ to complete the proof. ∎

The next result is somewhat more interesting and important, and is a good example of the power of Borell's inequality. The original proof of this result, which appears as Exercise 1.1, is due to Fernique (1978), and involves a comparatively long and sophisticated calculation.