II. TWO BASIC RESULTS

Despite the title of this Chapter, there are probably *three* basic results in the theory of Gaussian processes, that make this theory both manageable and special. The first is the existence theorem that to any positive semi-definite function R there corresponds a centered Gaussian process with covariance function R; an important, but not particularly exciting result.

The second is that the supremum of a Gaussian process behaves much like a single Gaussian variable with variance equal to the largest variance achieved by the entire process. In the way that we shall present it, this is Borell's inequality, and is the key to all results about Gaussian continuity, boundedness, and suprema.

The third is that if two centered processes have identical variances (i.e. $EX_t^2 = EY_t^2$ for all $t \in T$), but one process is more "correlated" than the other (i.e. if $EX_tX_s \geq EY_tY_s$ for all $s, t \in T$) then the more correlated process has the stochastically smaller maximum, in the sense that $P\{\sup X_t > \lambda\} \leq P\{\sup Y_t > \lambda\}$ for all $\lambda > 0$. This is Slepian's inequality, and without this result many of the most basic results in the theory of Gaussian processes would have no proof.

Both Borell's and Slepian's inequality are very special in that analagous results for non-Gaussian processes are extremely rare. (We shall see some exceptions to this rule later). The fact that even for Gaussian processes the $\sup X_t$ in Slepian's inequality cannot be replaced by as simple a variant as $\sup |X_t|$ is also indicative how very lucky we are that a result of this kind holds at all.

1. Borell's Inequality.

Let X be a centered Gaussian random variable with variance σ^2. Then choosing

$$\Psi(\lambda) = (2\pi)^{-\frac{1}{2}} \int_\lambda^\infty e^{-\frac{1}{2}x^2}\, dx,$$

to denote the standard Gaussian distribution function, straightforward approximations give that for all $\lambda > 0$

$$\begin{aligned} (1-\sigma^2\lambda^{-2})(\sigma/\sqrt{2\pi})\lambda^{-1}e^{-\frac{1}{2}\lambda^2/\sigma^2} &\leq P\{X > \lambda\} \\ &= \Psi(\lambda/\sigma) \\ &\leq (\sigma/\sqrt{2\pi})\lambda^{-1}e^{-\frac{1}{2}\lambda^2/\sigma^2} \end{aligned} \tag{2.1}$$

One immediate consequence of (2.1) is that

$$\lim_{\lambda\to\infty} \lambda^{-2} \log P\{X > \lambda\} = -(2\sigma^2)^{-1}. \tag{2.2}$$