Chapter 9. Models

Fitting models to data is a popular activity. For data taking values in a group or homogeneous space, the associated representation theory gives neat families of models. Briefly, if $P(x)$ is a probability on X, write $P(x) = e^{h(x)}$ with $h = \log P$. Then expand h in a natural basis b_i of $L(X)$: $h(x) = \Sigma\theta_i b_i(x)$. Any positive probability can be expanded in this way. Truncating the expansion to a fixed number of terms leads to families of simple measures or models; because b_i are orthogonal, θ_i are identifiable.

It is an interesting fact that many models introduced by applied workers fall into this class. The general story is presented first, then a specialization to data on spheres, then a specialization to partially ranked data. A brief review of other approaches to ranked data is followed by a section supplying the relevant exponential family theory.

A. Exponential families from representations

Let G be a group acting transitively on a compact set X. Let $L(X)$ denote the real valued continuous functions on X. Suppose X has an invariant distribution dx. The following abstracts an idea introduced by Lo (1977) and Beran (1979).

Definition. Let Θ be an invariant subspace of $L(X)$ containing the constants. Define a family of measures, one for each $\theta \in \Theta$, by specifying the densities to be

$$P_\theta(dx) = a(\theta)e^{\theta(x)}dx,$$

where $a(\theta)$ is a normalizing constant forcing $P_\theta(dx)$ to integrate to 1.

Suppose Θ is finite dimensional. Let b_0 =constant, $b_1, b_2, \ldots, b_p$ be a basis for Θ. Then the family can be parameterized as

$$(*) \qquad P_\theta(dx) = a(\theta)e^{\theta' b}dx, \ \theta \in \mathbb{R}^p, b = (b_1(x), \ldots, b_p(x)).$$

LEMMA 1. *The family * is well parameterized in the sense that $P_\theta = P_{\theta'}$ if and only if $\theta = \theta'$.*

Proof. Only the forward direction requires proof. If $P_\theta = P_{\theta'}$, then

$$(\theta - \theta') \cdot b(x) = \log(a(\theta')/a(\theta)) \text{ for all } x.$$

The left side is a linear combination of $b_1, b_2, \ldots, b_p$ which is constant. But $1, b_1, b_2, \ldots, b_p$ is a basis, so $\theta = \theta'$. □

In applications there is a decomposition into invariant subspaces $L(X) = V_0 \oplus V_1 \oplus V_2 \oplus \ldots$ and Θ's are chosen as a finite direct sum of subspaces. Usually