Chapter 8. Spectral Analysis

Often data are presented as a function $f(x)$ defined on some index set X. If X is connected to a group, the function f can be "Fourier expanded" and one may try to interpret its coefficients. This is a rich idea which includes the usual spectral analysis of time series and the analysis of variance.

This chapter develops the idea in stages. First, for data on groups (time series, binary vectors, and permutation data). Then the idea is developed for data on homogeneous spaces (the sphere and partially ranked data). Next some theory needed for practical computation is derived. All of this is illustrated for some classical designs in the analysis of variance. Finally, some research projects are spelled out.

A. Data on groups

1. *Time series analysis.* Fourier analysis of time series or other signals is a familiar scientific activity. For example, data on the number of babies born daily in New York City over a five year period are studied by Izenman and Zabell (1978). Here $X = \{1, 2, \ldots, n\}$ with $n = 5 \times 365 + 1$. The data are represented as a function $f(x) = \#$ born on day x. Inspection of this data shows strong periodic phenomena: about 450 babies are born on each week day and about 350 on each day of the weekend. (Physicians don't like to work on weekends.) There might also be monthly or quarterly effects.

To examine (and discover) such phenomena, scientists pass from the original data $f(x)$ to its Fourier transform

$$\hat{f}(y) = \Sigma_x f(x)\ e^{2\pi ixy/n}$$

where the sum runs over $x = 0, 1, \ldots, n-1$. Fourier inversion gives

$$f(x) = \frac{1}{n}\Sigma_y \hat{f}(y)\ e^{-2\pi ixy/n}.$$

It sometimes happens that a few values of $\hat{f}(y)$ are much larger than the rest and determine f in the sense that f is closely approximated by the function defined by using only the large Fourier coefficients in the inversion formula. When this happens, we have f approximated by a few simple periodic functions of x, e.g. $e^{-2\pi ixy/n}$, and may feel we understand the situation.

The hunting and interpretation of periodicities is one use of spectral analysis. A splendid introduction to this subject is given by Bloomfield (1976). A more advanced treatment from the same point of view is given by Brillinger (1975).