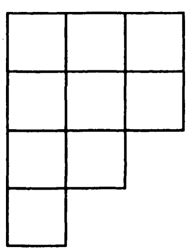
# Chapter 7. Representation Theory of the Symmetric Group

We have already built three irreducible representations of the symmetric group: the trivial, alternating and $n-1$ dimensional representations in Chapter 2. In this chapter we build the remaining representations and develop some of their properties.

To motivate the general construction, consider the space $X$ of the unordered pairs $\{i,j\}$ of cardinality $\binom{n}{2}$. The symmetric group acts on these pairs by $\pi\{i,j\} = \{\pi(i),\pi(j)\}$. The permutation representation generated by this action can be described as an $\binom{n}{2}$ dimensional vector space spanned by basis vectors $e_{\{i,j\}}$. This space splits into three irreducibles: A one-dimensional trivial representation is spanned by $\overline{v} = \Sigma\, e_{\{i,j\}}$. An $n-1$ dimensional space is spanned by $v_i = \Sigma_j\, e_{\{i,j\}} - c\overline{v}, 1 \leq i \leq n$, with $c$ chosen so $v_i$ is orthogonal to $\overline{v}$. The complement of these two spaces is also an irreducible representation. A direct argument for these assertions is given at the end of Section A. The arguments generalize. The following treatment follows the first few sections of James (1978) quite closely. Chapter 7 in James and Kerber (1981) is another presentation.

## A. Construction of the irreducible representations of the symmetric group.

There are various definitions relating to diagrams, tableaux, and tabloids. Let $\lambda = (\lambda_1,\ldots,\lambda_r)$ be a partition of $n$. Thus, $\lambda_1 \geq \lambda_2 \ldots \geq \lambda_r$ and $\lambda_1 + \ldots + \lambda_r = n$. The *diagram* of $\lambda$ is an ordered set of boxes with $\lambda_i$ boxes in row $i$. If $\lambda = (3,3,2,1)$, the diagram is



If $\lambda$ and $\mu$ are partitions of $n$ we say $\lambda$ *dominates* $\mu$, and write $\lambda \trianglerighteq \mu$, provided that $\lambda_1 \geq \mu_1, \lambda_1+\lambda_2 \geq \mu_1+\mu_2,\ldots$, etc. This partial order is widely used in various areas of mathematics. It is sometimes called the order of *majorization.* There is a book length treatment of this order by Marshall and Olkin (1979). They show that $\lambda \trianglerighteq \mu$ if and only if we can move from the diagram of $\lambda$ to the diagram of $\mu$ by moving blocks from the right hand edge upward, one at a time, such that at each stage the resulting configuration is the diagram of a partition. Thus, $(4,2) \triangleright (3,3)$, but $(3,3)$, and $(4,1,1)$ are not comparable. See Hazewinkel and Martin (1983) for many novel applications of the order.