Chapter 4. Probabilistic Arguments

A. Introduction — Strong uniform times.

There are a number of other arguments available for bounding the rate of convergence to the uniform distribution. This chapter discusses the method of strong uniform times and coupling. Let's begin with a simple example, drawn from Aldous and Diaconis (1986).

Example 1. Top in at random. Consider mixing a deck of n cards by repeatedly removing the top card and inserting it at a random position. This corresponds to choosing a random cycle:

$$(1) \qquad P(\text{id}) = P(21) = P(321) = P(4321) = \ldots = P(nn-1\ldots 1) = \frac{1}{n}.$$

The following argument will be used to show that $n \log n$ shuffles suffice to mix up the cards. Consider the bottom card of the deck. This card stays at the bottom until the first time a card is inserted below it. This is a geometric waiting time with mean n. As the shuffles continue, eventually a second card is inserted below the original bottom card (this takes about $n/2$ further shuffles). The two cards under the original bottom card are equally likely to be in relative order low-high or high-low.

Similarly, the first time a third card is inserted below the original bottom card, each of the six possible orders of the three bottom cards is equally likely. Now consider the first time T that the original bottom card comes to the top. By an inductive argument, all $(n-1)!$ arrangements of the lower cards are equally likely. When the original bottom card is inserted at random, all $n!$ possible arrangements of the deck are equally likely.

When the original bottom card is at position k from the bottom, the waiting time for a new card to be inserted is geometric with mean n/k. Thus the waiting time T has mean $n + \frac{n}{2} + \frac{n}{3} + \ldots + \frac{n}{n} \doteq n \log n$.

To make this argument rigorous, introduce strong uniform times. Let G be a finite group. Intuitively, a stopping time is a rule which looks at a sequence of elements in G and says "stop at the jth one." The rule is allowed to depend on what appears up to time j, but not to look in the future. Formally, *a stopping time* is a function $T: G^\infty \to \{1, 2, \ldots, \infty\}$ such that if $T(\underset{\sim}{s}) = j$ then $T(\underset{\sim}{s}') = j$ for all $\underset{\sim}{s}'$ with $s'_i = s_i$ for $1 \le i \le j$. Let Q be a probability on G, X_k the associated random walk, P the associated probability on G^∞. A *strong uniform time T* is a stopping time T such that for each $k < \infty$,

$$(2) \qquad P\{T = k,\ X_k = s\} \quad \text{is constant in } s.$$