Chapter 2. Basics of Representations and Characters

A. Definitions and examples.

We start with the notion of a *group*: a set G with an associative multiplication $s, t \to st$, an identity id, and inverses s^{-1}. A *representation* ρ of G assigns an invertible matrix $\rho(s)$ to each $s \in G$ in such a way that the matrix assigned to the product of two elements is the product of the matrices assigned to each element: $\rho(st) = \rho(s)\rho(t)$. This implies that $\rho(id) = I$, $\rho(s^{-1}) = \rho(s)^{-1}$. The matrices we work with are all invertible and are considered over the real or complex numbers. We thus regard ρ as a homomorphism from G to $GL(V)$ — the linear maps on a vector space V. The dimension of V is denoted d_ρ and called the *dimension of* ρ.

If W is a subspace of V stable under G (i.e., $\rho(s)W \subset W$ for all $s \in G$), then ρ restricted to W gives a *subrepresentation*. Of course the zero subspace and the subspace $W = V$ are trivial subrepresentations. If the representation ρ admits no non-trivial subrepresentation, then ρ is called *irreducible*. Before going on, let us consider an example.

Example. S_n the permutation group on n letters.

This is the group S_n of $1-1$ mappings from a finite set into itself; we will use the notation $\left[{1 \atop \pi(1)}\ {2 \atop \pi(2)} \cdots {n \atop \pi(n)}\right]$. Here are three different representations. There are others.

(a) The *trivial representation* is 1-dimensional. It assigns each permutation to the identity map $\rho(\pi)x = x$.

(b) The *alternating representation* is also 1-dimensional. To define it, recall the sign of a permutation π is $+1$ if π can be written as a product or an even number of transpositions $\pi = \overbrace{(ab)(cd)\ldots(ef)}^{\text{even \# of factors}}$. The sign of π is -1 if π can be written as an odd number of transpositions. Elementary books on group theory show that $\text{sgn}(\pi)$ is well defined and that $\text{sgn}(\pi_1\pi_2) = \text{sgn}(\pi_1)\text{sgn}(\pi_2)$. It follows that $x \to \text{sgn}(\pi) \cdot x$ is a 1-dimensional representation.

(c) The *permutation representation* is an n-dimensional representation. To define it, consider the standard basis $e_1, \ldots, e_n$ of $\mathbb{R}^n$. It is only necessary to define the linear map $\rho(\pi)$ on the basis vectors. Define $\rho(\pi)e_j = e_{\pi(j)}$. The matrix of a linear map L is defined by $L(e_j) = \Sigma L_{ij}e_i$. With this convention, $\rho(\pi)_{ij}$ is zero or one. It is one if and only if $\pi(j) = i$, so $\rho(\pi)_{ij} = \delta_{i\pi(j)}$. I will write permutations right to left. Thus $\pi_2\pi_1$ means first perform π_1 and then perform π_2.

We will also be using cycle notation for permutations, $(a_1a_2\ldots a_k)$ means $a_1 \to a_2$, $a_2 \to a_3 \ldots a_k \to a_1$. Thus $(1\ 2)(2\ 3) = (1\ 2\ 3)$ (and *not* $(1\ 3\ 2)$).