CHAPTER 2. ANALYTIC PROPERTIES

DIFFERENTIABILITY AND MOMENTS

The cumulant generating function has several nice properties. Among these are the fact that its defining expression may be differentiated under the integral sign. In this manner one obtains the moments of X from the derivatives of ψ.

One needs first to establish a simple bound.

2.1 Lemma

Let $B = \text{conhull}\ \{b_i : i=1,\ldots,I\} \subset R^k$. Let $C \subset B^\circ$ be compact and let $b_0 \in C$. Then there are constants K_ℓ (depending on C,B) $\ell=0,1,\ldots$ such th

$$(1) \qquad ||x||^\ell e^{b\cdot x} \leq K_\ell \sum_{i=1}^{I} e^{b_i \cdot x} \qquad \forall\ b \in C\ ,\quad x \in R^k \quad .$$

Also,

$$(2) \qquad \left| \frac{e^{b\cdot x} - e^{b_0\cdot x}}{||b - b_0||} \right| \leq K_1 \sum_{i=1}^{I} e^{b_i\cdot x} \quad , \quad b \in C\ , \quad x \in R^k \quad .$$

Proof. Let $\varepsilon > 0$. Note that there exists a $K_{\ell,\varepsilon} < \infty$ such that

$$|r|^\ell \leq K_{\ell,\varepsilon}\, e^{\varepsilon|r|} \qquad \forall \quad r \in R$$

since

$$\lim_{|r|\to\infty} |r|^\ell / e^{\varepsilon|r|} = 0 \quad .$$

Let $\{e_i : i=1,\ldots,k\}$ denote the elementary (orthogonal) unit vectors in R^k. Then

$$||x||^\ell \leq k^{(\ell-2)/2} \sum_{i=1}^{k} |x_i|^\ell \leq K'_{\ell,\varepsilon} \sum_{i=1}^{k} e^{\varepsilon|x_i|} < K'_{\ell,\varepsilon} \sum_{i=1}^{k} (e^{\varepsilon e_i\cdot x} + e^{-\varepsilon e_i\cdot x}),$$