LECTURE XIV. A THIRD ABSTRACT NORMAL APPROXIMATION THEOREM

In order to see the relation of the results of the tenth lecture to the abstract formulation of the first lecture it will be necessary to introduce fairly elaborate formalism. Roughly speaking, in order to obtain an exchangeable pair as in the first lecture we introduce a new random point conditionally independent of the original one given $\mathcal{C}$ and with the same conditional distribution given $\mathcal{C}$. This requires a new sample space, big enough to carry the original σ-algebra $\tilde{\mathcal{B}}$ and its copy $\tilde{\mathcal{B}}'$ corresponding to the new random point. The resulting structure seems quite formidable, but I believe it will be useful in the long run, although the simpler treatment of the tenth lecture and the even simpler treatment introduced at the end of the first lecture should suffice for many problems.

Let $(\tilde{\tilde{\Omega}},\tilde{\tilde{\mathcal{B}}},\tilde{\tilde{P}})$ be a probability space, let $\tilde{\mathcal{B}}$, $\tilde{\mathcal{B}}'$, and $\mathcal{C}$ be sub-σ-algebras of $\tilde{\tilde{\mathcal{B}}}$ and suppose that, under $\tilde{\tilde{P}}$, $\tilde{\mathcal{B}}$ and $\tilde{\mathcal{B}}'$ are conditionally independent given $\mathcal{C}$. Also let $\mathcal{B}$ be a sub-σ-algebra of $\tilde{\mathcal{B}}$ and $\mathcal{B}'$ a sub-σ-algebra of $\tilde{\mathcal{B}}'$, and let $\gamma\colon \tilde{\tilde{\Omega}} \to \tilde{\tilde{\Omega}}$ be an involution, that is

(1) $$\gamma^2 = I_{\tilde{\tilde{\Omega}}} ,$$

such that

(2) $$\gamma^{-1} B \in \tilde{\tilde{\mathcal{B}}} \quad \text{for all} \quad B \in \tilde{\tilde{\mathcal{B}}},$$

(3) $$B \in \tilde{\mathcal{B}} \Leftrightarrow \gamma^{-1} B \in \tilde{\mathcal{B}}',$$

(4) $$B \in \mathcal{B} \Leftrightarrow \gamma^{-1} B \in \mathcal{B}',$$