Lecture XII. RANDOM ALLOCATIONS

I shall derive three of the simpler results given by Kolchin, Sevastyanov, and Chistyakov in their book Random allocations, using the methods introduced in the first lecture. One is the exact joint distribution of the numbers of urns containing 0,1,2,... balls when ν balls are distributed at random (uniformly) among k urns. The second is the analogous problem for cycle lengths of random permutations. The third is the Poisson approximation to the distribution of the number of empty urns in the first problem, when the expected proportion of empty urns is small.

In the urn problem, let N_α be the number of urns containing α balls for $\alpha \in \{0,\ldots,\nu\}$. We want to compute the

$$p(n) = P\{N = n\} \tag{1}$$

for $n: \{0,\ldots,\nu\} \to Z^+$ satisfying

$$\Sigma n_\alpha = k \tag{2}$$

and

$$\Sigma \alpha n_\alpha = \nu. \tag{3}$$

Of course, for n that do not satisfy (2) and (3), p(n) = 0. We shall see that

$$p(n) = \frac{1}{\Pi[(\alpha!)^{n_\alpha} n_\alpha!]} \frac{k!\nu!}{k^\nu}. \tag{4}$$

In order to study the distribution of N we define a new random vector N', obtained by removing a randomly selected ball (uniformly distributed over the