LECTURE VIII. POISSON APPROXIMATIONS

An example of approximation by the Poisson distribution has already been given in the seventh lecture. Here I shall discuss this subject in the context of the abstract formalism of the first lecture, with special emphasis on the classical problem of the total number of occurrences of a large number of independent random events with small probabilities. Most of this work was done by Chen (1975a).

Theorem 1: In order that the random variable W taking values in Z^+, the set of all non-negative integers, have a Poisson distribution with parameter λ it is necessary and sufficient that, for all bounded functions $f: Z^+ \to R$,

(1) $$E[\lambda f(W+1) - Wf(W)] = 0.$$

Proof of necessity: Suppose W has a Poisson distribution with parameter λ, that is, for all $w \in Z^+$

(2) $$P\{W=w\} = e^{-\lambda} \frac{\lambda^w}{w!}.$$

Then, for all bounded $f: Z^+ \to R$

(3) $$EWf(W) = e^{-\lambda} \sum_{w=0}^{\infty} wf(w) \frac{\lambda^w}{w!}$$

$$= e^{-\lambda}\lambda \sum_{w'=0}^{\infty} f(w'+1) \frac{\lambda^{w'}}{w'!} = \lambda Ef(W+1).$$

Observe that the value of f(0) is irrelevant to this result. Of course the identity (3) does not really require f to be bounded. It is valid if the