LECTURE IV. THE NUMBER OF ONES IN THE BINARY EXPANSION OF A RANDOM INTEGER

Let n be a natural number and X a random variable uniformly distributed over the set $\{0,\ldots,n-1\}$. We shall see that for large n the number of ones in the binary expansion of X has approximately a binomial distribution, the distribution of the number of successes in k independent trials with probability one-half, where k is determined by

(1) $$2^{k-1} < n \leq 2^k.$$

The expected value of the number of ones in this expansion was studied as a function of n by Delange (1975). In Diaconis (1977) the present problem was studied by the method of the third lecture. Here I shall give a slightly different treatment in order to emphasize the notion of approximation by the binomial distribution rather than the asymptotically equivalent normal distribution. At the end of the lecture I shall also sketch a proof of the same result by an induction argument, not related to the main ideas of this series of lectures.

Let n be a natural number and X a random variable uniformly distributed over the set $\{0,\ldots,n-1\}$. For the binary expansions of n-1 and X, I shall write

(2) $$a = n-1 = \sum_{i=1}^{k} a_i 2^{k-i},$$

and

(3) $$X = \sum_{i=1}^{k} X_i 2^{k-i}.$$