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A MEASURE OF THE CONFORMITY OF A PARAMETER SET TO A TREND: THE PARTIALLY ORDERED CASE[1]

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Inferences concerning order restrictions on a collection of parameters, $\theta_1, \theta_2, \ldots, \theta_k$, are considered with the order restrinctions of the form, $\theta_i \leq \theta_j$ for $i \preceq j$ where $\preceq$ is a partial order on $1, 2, \ldots, k$. Clearly, some parameter sets conform more closely to these order restrinctions than others. We are interested in measures of the degree of conformity. Some of the measures available in the literature for the totally ordered case are generalized to the partially ordered case and the theory developed is applied in several tests of order restricted hypotheses.

1. Introduction. In various situations, one is interested in a collection of parameters $\theta_1, \theta_2, \ldots, \theta_k$ which are believed to satisfy certain known order restrictions and inference procedures which make use of this ordering information are preferred. We consider order restrictions that are induced by partial orders on $\Omega = \{1, 2, \ldots, k\}$. That is, suppose that $\preceq$ is a partial order on Ω and that the order restrictions are $\theta_i \leqslant \theta_j$ when $i \preceq j$. Such a vector $\theta = (\theta_1, \theta_2, \ldots, \theta_k)$ is said to be isotone (with respect to $\preceq$). In studying such inference procedures it is helpful to have a measure of the degree of conformity to the order restrictions. For instance, a test of H_0: θ is constant versus H_1: θ is isotone, but not constant should have power that increases with the degree of conformity. For a non-simple null hypothesis such a concept could be useful in identifying a least favorable configuration. In a Bayesian approach, priors which assign larger probabilities to parameters conforming more closely to the order restrictions would be sought.

Barlow, Bartholomew, Bremner and Brunk (1972) contains a thorough discussion of order restricted inference. Robertson and Wright (1982) develop several measures of conformity for the totally ordered case, ie. $\theta_1 \leqslant \theta_2 \leqslant \ldots \leqslant \theta_k (1 \preceq 2 \preceq \ldots \preceq k)$. In considering unimodal structures, partial orders of the type $1 \preceq 2 \preceq \ldots \preceq r \succeq r + 1 \succeq \ldots \succeq k$ arise and when making one-sided comparisons of several treatments with a common control, the partial order $1 \preceq i$ for $i = 2, 3, \ldots, k$ occurs. (See Bartholomew (1959) and Robertson and Wright (1981).) Suppose that a dependent variable has mean $\theta(i,j)$ when the first independent variable is fixed at level i, $1 \leqslant i \leqslant r$, and the second independent variable is fixed at level j, $1 \leqslant j \leqslant c$. If the levels are increasing and if $\theta(\cdot,\cdot)$ increases with each independent variable as the other is held fixed, then the order restrictions are $\theta(i,j) \leqslant \theta(s,t)$ for $i \leqslant s$ and $j \leqslant t$. This is another example of a partial order that is not total. We extend the measures of conformity in Robertson and Wright (1982) to the partially ordered case.

A set $L \subset \Omega$ is a lower layer provided $i \in L$ whenever $i \preceq j$ and $j \in L$. We denote the collection of lower layers by $\mathcal{L}$. To allow for different weights on the parameters, let $\mathbf{w}$ be a positive weight function defined on Ω, ie. $\mathbf{w} = (w_1, w_2, \ldots, w_k)$. For situations in which the degree of conformity should be translation invariant, we consider the relationship

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