Inequalities in Statistics and Probability
IMS Lecture Notes–Monograph Series Vol. 5 (1984), 228–235

DUAL CONVEX CONES OF ORDER RESTRCITIONS WITH APPLICATIONS[1]

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The concept of closed convex cones in finite dimensional Euclidian space and their duals has proven to be a useful construct. Here dual cones are exhibited for specific closed, convex cones including those pertaining to starshaped orderings and concave (convex) functions.

Applications include finding projections involving starshaped orderings, generalizations of Chebyshev's (Kimball's) inequality, an inequality for concave (convex) functions and a characterization of certain kinds of positive dependence.

1. Introduction. Several authors have made extensive use of the concept of convex cones and their duals in $\mathcal{R}^n$. Among these are Rockafellar (1970), Robertson and Wright (1981), and Barlow and Brunk (1972). Here we wish to specifically exhibit certain convex cones and their duals and discuss the implications.

To be precise, we call $K \subset R^n$ a convex cone if (a) $\mathbf{x},\mathbf{y} \in K \Rightarrow \mathbf{x} + \mathbf{Y} \in K$, and (b) $\mathbf{x} \in K$, $a \geqslant 0 \Rightarrow a\,\mathbf{x} \in K$. Of course if K is a convex cone, so is $-K = \{\mathbf{x}: -\mathbf{x} \in K\}$ which we will call the "negative" of K.

Another important convex cone induced by K is the "dual" of K. For a fixed positive vector $\mathbf{w}$, the dual of K is given by

$$K^{\mathbf{w}*} = \{\mathbf{y}:(\mathbf{x},\mathbf{y}) = \Sigma_{i=1}^{n} x_i y_i w_i \leqslant 0 \text{ for all } \mathbf{x} \in K\}.$$

(Some authors prefer the term "polar" to "dual." Some also define the dual as the negative of our dual.) Of course if K is closed, then $(K^{\mathbf{w}*})^{\mathbf{w}*} = K$. It is evident that if $K_1 \subset K_2$, then $K_1^{\mathbf{w}*} \supset K_2^{\mathbf{w}*}$.

New convex cones can be formed from existing cones in several ways. Two important methods are through intersections and direct sums.

If the closed, convex cones $K_1, \ldots, K_n$ are sufficiently nice (say finitely generated), the direct sum $\Sigma_{i=1}^{n} K_i = \{\Sigma_{i=1}^{n} \mathbf{x}_i | \mathbf{x}_i \epsilon K_i,\ i=1, \ldots, n\}$ is also a closed, convex cone. However, in general the closure property is not guaranteed (see Hestenes (1975), pp. 196–198). Nevertheless, intersections and direct sums of closed, convex cones are closely related because it is always true that $(\Sigma_{i=1}^{n} K_i)^{\mathbf{w}*} = \bigcap_{i=1}^{n} K_i^{\mathbf{w}*}$ and

$$(\bigcap_{i=1}^{n} K_i)^{\mathbf{w}*} = \Sigma_{i=1}^{n} K_i^{\mathbf{w}*} \tag{1.1}$$

if the latter cone is closed. This is guaranteed if the relative interiors of the K_i have a point in common (see Rockafellar (1970), p. 146) or, as we said, if the $K_i^{\mathbf{w}*}$ are finitely generated. (1.1) is equivalent to the well-known Farkas' Lemma if the K_i are generated by a single vector.

An important cone, especially in the area of isotone regression, is the cone of vectors which are nondecreasing, i.e.

$$K_I = \{\mathbf{x} | x_1 \leqslant x_2 \leqslant \ldots \leqslant x_n\}. \tag{1.2}$$

The dual cone here, as discussed in Barlow and Brunk (1972), is

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[1] This work was partially supported by ONR Contract N00017–78–C–0655.

AMS 1980 subject classifications. Primary 52A40, secondary 52A20.

Key words and phrases: Convex cones, dual cones, polars, projections, starshaped orderings, concave (convex) functions, Chebyshev's (Kimball's) Inequality, positively dependent random variables, order restrictions.