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APPLICATIONS OF A UNIFIED THEORY OF MONOTONICITY IN SELECTION PROBLEMS

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In this paper, the general monotonicity results concerning selection problems derived by Berger and Proschan (1984) are reviewed. They are then applied to several different formulations of the selection problem. These include comparison with a control and restricted subset selection problems. Several classes of selection rules previously proposed in the literature are shown to possess the monotonicity properties. In addition, a new class of rules for the restricted subset selection formulation is proposed and shown to possess the monotonicity properties.

1. Introduction. In this paper we study some monotonicity properties of ranking and selection rules.

Recall that in a selection problem the general goal is to determine which of several populations possesses the largest value of some parameter. Based on random observations from the populations, a selection rule selects a subset of the populations and leads to an assertion such as, "The population with the largest parameter is in the selected subset." (Different formulations of the selection problem entail different assertions resulting from the selection rule.) A reasonable selection rule should be more likely to choose populations with larger parameters rather than populations with smaller parameters. This property of selection rules is called monotonicity.

In this paper we study some general monotonicity properties of a broad class of selection rules in a unified manner. We also discuss applications of these general results to several different formulations of the selection problem.

In symbols, let $\mathbf{X} = (X_1, \ldots, X_n)$ be a random observation with distribution $F(\mathbf{x}; \lambda)$, where the unknown parameter vector $\lambda = (\lambda_1, \ldots, \lambda_n) \in \Lambda \subset \mathscr{R}^n$. The general goal of a selection problem is to decide which of the coordinates of λ are the largest or which are larger than a value λ_0 (possibly unknown). A (nonrandomized) selection rule $S(\mathbf{x})$ is any measurable mapping from the sample space $\mathcal{X}$ of $\mathbf{X}$ into the set of subsets of $\{1, \ldots, n\}$. Having observed $\mathbf{X} = \mathbf{x}$, the selection rule S asserts that the largest parameters are in $\{\lambda_i : i \in S(\mathbf{x})\}$. The subset $S(\mathbf{X})$ may be of fixed or random size depending on the formulation of the selection problem under consideration. See, for example, Bechhofer (1954) (fixed size), Gupta and Sobel (1958) (random size), and Gupta (1965) (random size).

Gupta (1965) calls a selection rule *montone* if

$$\lambda_i \geq \lambda_j \text{ implies } P_\lambda(i \in S(\mathbf{X})) \geq P_\lambda(j \in S(\mathbf{X})). \tag{1.1}$$

Monotonicity is a desirable property of a selection rule since the selected subset is supposed to consist of the large values of λ_i. On a case by case basis, various authors have shown that their proposed selection rules are monotone. Monotonicity has not been investi-

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