Inequalities in Statistics and Probability
IMS Lecture Notes–Monograph Series Vol. 5 (1984), 173–177

STOCHASTIC MAJORIZATION OF THE LOG-EIGENVALUES OF A BIVARIATE WISHART MATRIX[1]

BY MICHAEL D. PERLMAN
University of Washington

Let $l = (l_1,l_2)$ and $\lambda = (\lambda_1,\lambda_2)$, where $\lambda_1 \geq \lambda_2 > 0$ are the ordered eigenvalues of $\mathbf{S}$ and Σ, respectively, and $\mathbf{S} \sim W_2(n,\Sigma)$ is a bivariate Wishart matrix. Let $\mathbf{m} = (m_1,m_2)$ and $\mu = (\mu_1,\mu_2)$, where $m_i = \log l_i$ and $\mu_i = \log \lambda_i$. It is shown that $P_\mu\{\mathbf{m} \notin B\}$ is Schur-convex in μ whenever B is a Schur-monotone set, i.e. $[\mathbf{x} \in B,\ \mathbf{x}$ *majorizes* $\mathbf{x}^*] \Rightarrow \mathbf{x}^* \in B$. This result implies the unbiasedness and power-monotonicity of a class of invariant tests for bivariate sphericity and other orthogonally invariant hypotheses.

1. Introduction. Let $\mathbf{S} \sim W_2(n,\Sigma)$ be a bivariate Wishart matrix with n degrees of freedom ($n \geq 2$) and expected value $n\Sigma$ (Σ positive definite). We shall be concerned with the power functions of orthogonally invariant tests for invariant testing problems such as the following:

$$(1.1)\qquad \begin{aligned} &H_{01}: \Sigma = \sigma^2\mathbf{I}, \sigma^2 \textit{ arbitrary} && vs.\ K_1: \Sigma \textit{ arbitrary} \\ &H_{02}: \Sigma = \mathbf{I} && vs.\ K_2: \Sigma \textit{ arbitrary} \\ &H_{03}: \Sigma = \mathbf{I} && vs.\ K_3: \Sigma - \mathbf{I} \textit{ positive definite} \\ &H_{04}: \Sigma = \mathbf{I} && vs.\ K_4: \Sigma - \mathbf{I} \textit{ negative definite}. \end{aligned}$$

Orthogonally invariant tests depend on $\mathbf{S}$ only through $l = (l_1, l_2)$, where $l_1 \geq l_2 (>0)$ are the ordered eigenvalues of $\mathbf{S}$. Because the power functions of such tests depend on Σ only through $\lambda = (\lambda_1,\lambda_2)$, where $\lambda_1 \geq \lambda_2$ (> 0) are the ordered eigenvalues of Σ, we may assume throughout this paper that $\Sigma = \mathbf{D}_\lambda \equiv diag(\lambda_1,\lambda_2)$.

The notions of majorization and Schur-convexity play an important role in determining such properties as unbiasedness and power monotonicity of invariant tests. To illustrate, consider the likelihood ratio test (LRT) for testing H_{01} (bivariate sphericity) *vs.* K_1. The acceptance region can be expressed in the equivalent forms

$$(1.2)\qquad \{\mathbf{S} | tr\mathbf{S}/|\mathbf{S}|^{1/2} \leq c\} \Leftrightarrow \{l | (l_1 + l_2)/(l_1 l_2)^{1/2} \leq c\}.$$

Since

$$(1.3)\qquad tr\mathbf{S}/|\mathbf{S}|^{1/2} = (s_{11} + s_{22})/((s_{11}s_{22})^{1/2}|\mathbf{R}|^{1/2}) = (e^{t_1}+e^{t_2})/(e^{(t_1+t_2)/2}|\mathbf{R}|^{1/2}),$$

where $\mathbf{S} = (s_{ij})_{i,j=1,2}$, $\mathbf{R}$ is the sample correlation matrix, and $t_i = \log s_{ij}$, and since s_{11}, s_{22}, and $\mathbf{R}$ are independent with $s_{ii} \sim \lambda_i\chi_n^2$ when $\Sigma = D_\lambda$, conditioning on $\mathbf{R}$ reduces the problem to the study of the power function of the LRT for equality of scale parameters ($\lambda_1 = \lambda_2$) based on the independent χ^2-variates s_{11} and s_{22} with equal degrees of freedom. It is easy to show that the joint density of $\mathbf{t} \equiv (t_1,t_2)$ is Schur-concave (in fact, permutation-invariant and log concave) with location parameter $\mu \equiv (\mu_1,\mu_2) \equiv (\log \lambda_1, \log \lambda_2)$, and that for fixed $\mathbf{R}$ the region

[1] This research was supported in part by National Science Foundation Grants MCS 80–02167 and MCS 83–01807.
AMS 1980 subject classifications. Primary 62H15; Secondary 62H10.
Key words and phrases: Bivariate Wishart distribution, log-eigenvalues, stochastic majorization, Schur function.