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REGIONS WHOSE PROBABILITIES INCREASE WITH THE CORRELATION COEFFICIENT AND SLEPIAN'S THEOREM

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Let **X** have a multivariate normal distribution. Slepian (1962) proved that the upper and lower orthants ($\mathbf{x} \leq \mathbf{c}$) and ($\mathbf{x} \geq \mathbf{c}$) have the property that their probabilities are nondecreasing in each ρ_{ij}. This easily implies, in the bivariate case, that if $A = Q_1 \cup Q_3 \cup B$, where Q_1 is an upper quadrant, Q_3 is a lower quadrant, B is a disjoint union of horizontal or vertical infinite strips and the interiors of Q_1, Q_3 and B are disjoint, then $P(A)$ is nondecreasing in ρ. This paper shows that, within a broad class of bivariate regions, sets A of the type described above are the only sets whose probabilities increase with the correlation coefficient when the means and the variances of X_1, X_2 take arbitrary values. Some results are also given for the cases where the means and the variances are restricted in some way.

1. Introduction. Let $\mathbf{X} = (X_1, \ldots, X_n)$ have the multivariate normal distribution with mean vector μ, variance vector σ^2 and correlation matrix (ρ_{ij}). Slepian (1962) proved that certain orthant probabilities are nondecreasing in each ρ_{ij} separately. This result and its generalizations have several applications; see, for example, Slepian (1962), Šidák (1968) and Joag-dev, Perlman and Pitt (1983). It is natural to ask whether there are sets other than orthants whose probabilities are nondecreasing in each ρ_{ij}. In this paper, we deal mainly wth the bivariate case and obtain a result which can be considered as a partial converse to Slepian's result.

Following the number of the quadrants in the plane, we denote by Q_1 an upper quadrant of the type $x_1 \geqslant a_1$, $x_2 \geqslant a_2$. A lower quadrant will be denoted by Q_3. The term infinite horizontal strip will mean a set defined by $-\infty < x_1 < \infty$, $a_2 \leqslant x_2 \leqslant b_2$. An infinite vertical strip is defined similarly. We note that the probability of an infinite horizontal or vertical strip is constant in ρ. Therefore, the following corollary of Slepian's result is immediate. For ease of reference, we state it as a theorem.

SLEPIAN'S THEOREM. *Let $A \subset \mathscr{R}^2$ have the form $A = Q_1 \cup Q_3 \cup B$, where B is a finite disjoint union or horizontal (or vertical) infinite strips and the interiors of Q_1, Q_3 and B are disjoint. Then $P(A)$ is nondecreasing in ρ.*

In section 2, we show that, within a broad class of bivariate regions, the sets A described in Slepian's theorem are the only sets whose probabilities are nondecreasing in ρ, when the means μ_1, μ_2 and the variances σ_1^2, σ_2^2 are allowed to take arbitrary values. Such a result can be considered to be a partial converse to Slepian's theorem. When the means and variances are restricted in some way, it is possible to obtain some additional regions whose probabilities increase with ρ. Some results in this direction are given in Section 3. In Section 4, we discuss an equivalent form of Slepian's result in terms of covariances and show that its generalization based on the concept of association fails.

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