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PROBABILISTIC ORDERING OF SCHEFFÉ POLYHEDRA

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Some inequalities for spherically symmetric distributions are discussed using simple ideas from convex geometry. There are two dual orderings of the size of convex polytopes with respect to "width" in a random direction. One is equivalent to the ordering of content with respect to all spherically symmetric distributions. The other is the stochastic version of the mean width of convex geometry. Dual versions of known results are given and in particular the complete classification of the Platonic solids is listed. Some remarks are made about future developments.

1. Introduction. There is a duality between a problem in statistics of ordering certain regions with respect to probability content and an ordering based on the support function of a convex set. Recent results are discussed in the light of this connection.

The first ordering arises naturally in the statistical theory of multiple comparisons and by now has a considerable literature. Let $\mathcal{C}$ be a class of sets in p-dimensional Euclidean space $\mathcal{E}^p$. Let F be a family of probability measures on $\mathcal{E}^p$ with respect to which every member of $\mathcal{C}$ is measurable. We say that for two members C_1 and C_2 of $\mathcal{C}$, $C_1 > C_2$ if $\mu(C_1) > \mu(C_2)$ for all μ in F. It is usual to specialize $\mathcal{C}$ and F in various ways. Typically C may comprise all convex radially symmetric sets ($x \in C$ implies $-x \in C$) and F may be all unimodal, spherically symmetric distributions or their multivariate normal versions. Much of this material is summarised in Tong (1980).

In this paper we first restrict $\mathcal{C}$ to all closed star-shaped regions: $x \in C$ implies $\lambda x \in C$ for all $0 \leqslant \lambda \leqslant 1$. Thus C contains all points on the ray to the boundary point in any direction. Let F consist of all spherically symmetric distributions: all measures preserved under any rotation about the origin. We refer to the induced ordering as $>_h$. The following simple geometric characterisation comes as Theorem 1 in Bohrer and Wynn (1982). Let s be a random direction in $\mathcal{E}^p$ which may be interpeted as a point distributed with the uniform distribution on the surface of the unit sphere S_{p-1} in $\mathcal{E}^p$. Let $h(C,s)$ be the distance to the boundary of C from the origin in the direction s. Then the result is that, for C_1 and C_2 in $\mathcal{C}$, $C_1 >_h C_2$ if an only if $h(C_1,s) > h(C_2,s)$, where $>$ is stochastic ordering:

$$P[h(C_1,s) \geqslant r] > P[h(C_2,s) \geqslant r] \qquad \text{for all } 0 \leqslant r \leqslant \infty.$$

The proof follows directly from the fact that it is sufficient to prove that the $p-1$ dimensional area of the intersection with the spherical shell rS_{p-1} of radius r is at least as great for C_1 as for C_2, for all $0 \leqslant r \leqslant \infty$. Then since C_1 and C_2 are star-shaped these intersections are (proportional to) the s-probability of the boundary in the direction s lying outside or on rS_{p-1}.

Measures of the size of convex bodies abound in the field of convex geometry which has had a resurgence in recent years but has been little used in the field of multiple comparisons. The subject arises as a foundation for Minkowski's geometry of numbers and in particular for his theorem on the volume of n-dimensional lattices (see Stewart and Tall (1979) for an elementary treatment). One arm of the subject is loosely called integral geometry

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