Inequalities in Statistics and Probability
IMS Lecture Notes–Monograph Series Vol. 5 (1984), 141–149

STOCHASTIC ORDERING OF SPACINGS FROM DEPENDENT RANDOM VARIABLES

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Spacings (that is, the differences between successive order statistics) are useful in various applications in statistics. Many properties of the spacing are known when the spacings are constructed from a collection of independent identically distributed (i.i.d.) random variables. In this paper we study the spacings constructed from not necessarily i.i.d. random variables. We introduce models for which two sets of spacings, constructed from two sets of dependent random variables, can be stochastically ordered. Various examples will be given and applications for goodness-of-fit tests, tests for independence, density estimation and tests for outliers will be discussed.

1. Introduction. Let $\mathbf{X} = (X_1, \dots, X_n)$ denote an n-dimensional random vector and let

$$X_{(1)} \leq \dots \leq X_{(n)}$$

be the ordered components (order statistics) of $\mathbf{X}$. The nonnegative random variables

$$U_i = X_{(i+1)} - X_{(i)},\ i = 1, \dots, n-1$$

are called the *spacings* and have various applications in statistics. For example, certain nonparametric test procedures depend on the maximum spacing or on linear combinations of spacings (see, e.g., Pyke (1965), Weiss (1965), Rao and Sethuraman (1970) and Kirmani and Alam (1974)); certain estimation and test procedures based on order statistics, such as those which depend on the range or midrange, involve spacings (David (1970), Ch. 6); and certain tests for slippage (Karlin and Truax (1960)) and outliers (Barnett and Lewis (1978), Ch. 3) also depend on spacings. For a comprehensive treatment of spacings see Pyke (1965, 1972).

In the literature the problem of spacings has been treated extensively under the assumption that $X_1, \dots, X_n$ are independent and identically distributed (i.i.d.) random variables. In certain applications which involve a mixture of experiments, a (random) change of scale or a random shift in location may take place; then the random variables $X_1, \dots, X_n$ are no longer independent. In this paper we study how the degree of dependence affects the distribution of the spacings. In the case when $X_1, \dots, X_n$ are interchangeable, it follows from our main result that (in the model under consideration) the spacings vector $\mathbf{U} = (U_1, \dots, U_{n-1})$ becomes stochastically smaller if $X_1, \dots, X_n$ are more positively dependent (that is, when $X_1, \dots, X_n$ have more tendency to "hang together").

After stating the model and proving the main result in Section 2, we apply the result to an additive, a multiplicative and a ratio model. In Section 3, after combining results given in Shaked and Tong (1985), we obtain a partial ordering property for the spacings which correspond to a number of important multivariate distributions, such as the multivariate normal, multivariate stable, multivariate beta and the Dirichlet distribution. For all these distributions the corresponding spacings vector $\mathbf{U}$ can be partially ordered through the degree of dependence of the components $X_1, \dots, X_n$ of $\mathbf{X}$.

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[1] Research partially supported by the NSF Grant MCS-8200098.
[2] Research partially supported by the NSF Grant MCS-8100775.
AMS 1980 subject classifications. Primary 60E15; secondary 62G30.
Key words and phrases. Stochastic ordering of spacings, dependent random variables.