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ASYMPTOTIC INDEPENDENCE AND LIMIT THEOREMS FOR POSITIVELY AND NEGATIVELY DEPENDENT RANDOM VARIABLES[1]

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For random variables which are associated or which exhibit certain related types of positive and negative dependence, the independence structure is largely determined by the covariance structure. We survey results of this sort with particular emphasis on limit theorems for partial sums of stationary sequences.

1. Introduction. The purpose of this paper is to survey a number of results concerning the degree to which the independence structure is determined by the covariance structure for families of random variables which exhibit certain types of positive or negative dependence. The original such result is due to Lehmann (1966). We first recall Lehmann's definition of positive and negative quadrant dependent (PQD and NQD) random variables. X_1 and X_2 are said to be PQD if

$$(1.1) \quad H_{1,2}(x_1,x_2) \equiv P[X_1>x_1,X_2>x_2] - P[X_1>x_1]\,P[X_2>x_2] \geq 0 \text{ for all } x_1,x_2 \in \mathcal{R};$$

They are said to be NQD if X_1 and $(-X_2)$ are PQD.

Note that an equivalent condition to (1.1) is that $\mathrm{Cov}(f(X_1), g(X_2)) \geq 0$ for all real increasing (i.e. nondecreasing) f and g (such that $f(X_1)$ and $g(X_2)$ have finite variance). *In the following statement of Lehmann's result and throughout the rest of the paper we will assume, unless otherwise mentioned, that all random variables have finite variance.*

THEOREM 1 *(Lehmann (1966)). If X_1 and X_2 are PQD or NQD, then they are independent if and only if* $\mathrm{Cov}(X_1,X_2) = 0$.

Proof. This theorem is an immediate consequence of the identity (obtained from integration by parts),

$$(1.2) \qquad \mathrm{Cov}(X_1,X_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} H_{1,2}(x_1,x_2)\,dx_1 dx_2$$

and the pointwise positivity (resp. negativity) of $H_{1,2}$ for PQD (resp. NQD) variables. □

The results which we discuss in this paper concern multivariate generalizations of Theorem 1 of two types. The first type is a direct generalization in which joint uncorrelatedness implies joint independence. The second type is an indirect generalization in which approximate uncorrelatedness implies approximate independence in a sufficiently quantitative sense to lead to useful limit theorems for sums of dependent variables. In Section 2, we review all the results of the first type along with an ergodicity result of the second type; with one exception, these are based on inequalities for *distribution* functions. In Section 3, we review a number of results of the second type, including a triangular array limit theorem and a central limit theorem; these are based on inequalities for *characteristic* functions. In Section 4, we present some recent results which extend the inequalities and limit

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