Inequalities in Statistics and Probability
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ON CHEBYSHEV'S OTHER INEQUALITY

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We formulate the notion of a best possible inequality. This involves finding the largest class of functions and measures for which an inequality is true. We give two examples of Chebyshev's inequality, e.g. $\int_a^b d\mu \int_a^b fg d\mu \geqslant \int_a^b f d\mu \int_a^b g d\mu$ for all pairs (f,g) which are increasing if and only if $\int_a^x du \geqslant 0$, $\int_x^b du \geqslant 0$ for all x. Other examples include Jensen's inequality.

1. Introduction. Let μ be a probability measure on the real line and f and g increasing functions. Then

$$\int_{\mathcal{R}} fg \, d\mu \geqslant \int_{\mathcal{R}} f d\mu \int_{\mathcal{R}} g \, d\mu \tag{1.1}$$

says that the random variables f and g are positively correlated. This is Chebyshev's 'other' inequality.

It is common to ask if an inequality is best possible. In most instances this means having the largest (or smallest) constant(s) for which the inequality holds and settling the cases of possible equality. For (1.1) equality holds if one of the functions is a constant or the measure is a point mass.

In this paper we would like to explore a different meaning of 'best possible.' In order to formulate our ideas in the context of inequality (1.1), consider a related version

$$\int_a^b d\mu \int_a^b fg \, d\mu \geqslant \int_a^b f d\mu \int_a^b g \, d\mu \tag{1.2}$$

where $[a,b]$ is any real interval. It was already observed by Andreief (1883), that (1.2) holds under the hypothesis that

$$[f(x)-f(y)][g(x)-g(y)] \geqslant 0 \quad \text{for all } (x,y) \in [a,b] \times [a,b], \tag{1.3}$$

and

$$\mu \text{ is a non-negative measure.} \tag{1.4}$$

The condition (1.3) is read "f and g are similarly ordered," see Hardy, Littlewood, and Pólya ((1952), p. 43). (More history of the inequality (1.2) appears in the article by Mitrinović and Vasić (1974).) It is clear that (1.3) is satisfied if both f and g are increasing.

Our viewpoint is that the inequality (1.2) has "two variables," the pairs of functions and the measures. 'Best possible' should mean that:

(A) the inequality (1.2) holds for all similarly ordered pairs if and only if μ is a non-negative measure, and

(B) the inequality (1.2) holds for all non-negative measures if and only if f and g are similarly ordered.

We will show below that both statements are correct. This means that each class, similarly ordered pairs, and non-negative measures, is the largest class for which the inequality can be proved, given that it must hold for all elements in the other class.

Contrast this with the condition

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