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AN EXPANSION FOR SYMMETRIC STATISTICS AND THE EFRON-STEIN INEQUALITY

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The Efron-Stein inequality and a generalization by Bhargava are derived using a tensor-product basis and bounds for covariances of related symmetric statistics.

1. Introduction. Let $S(X_1, \ldots, X_n)$ be a symmetric function of its iid arguments. Its variance can be estimated by the jackknife technique as follows: assuming an augmented iid collection $X_1, \ldots, X_n, X_{n+1}$, form $S_i = S(X_1, \ldots, X_{i-1}, X_{i+1}, \ldots, X_{n+1})$, $i = 1, \ldots, n+1$ and $\bar{S} = (n+1)^{-1}\Sigma_{i=1}^{n+1} S_i$. Then $\mathrm{Var}\, S(X_1, \ldots, X_n)(=\mathrm{Var}\, S_i)$ is estimated by $Q = \Sigma_{i=1}^{n+1} (S_i - \bar{S})^2$. As part of an extensive study, Efron and Stein (1981) showed that Q is necessarily positively biased, an observation that has come to be known as the *Efron-Stein inequality*.

THEOREM 1.

$$\mathrm{Var}\, S(X_1, \ldots, X_n) \leq EQ \tag{1.1}$$

with equality iff S is linear in functions of its individual arguments.

Other proofs and extensions have been given by Bhargava (1980) and Karlin and Rinott (1982), and the inequality has already had interesting applications (Hochbaum and Steele (1982), Steele (1981), Steele (1982)). Our purpose here is to derive the inequality by using an idea exploited for other purposes in Rubin and Vitale (1980): expansion of symmetric statistics in a tensor-product basis. The approach yields attractive, concrete representations and is particularly well-adapted to proving the E-S inequality by first establishing a universal bound on the covariance of related symmetric statistics. It is an alternative to the ANOVA-type expansions used elsewhere.

2. The Efron-Stein Inequality via Covariance Bounds. If $e_0(X_1) \equiv 1$, $e_1(X_1)$, $e_2(X_1)$, ... form an orthonormal basis for the square integrable functions of X_1, then products of the type $\Pi_{i=1}^n e_{\nu_i}(X_i)$ form an orthonormal basis for the square integrable functions of $\mathbf{X} = (X_1, \ldots, X_n)$. For ease of notation we denote the above product by $e_\nu(\mathbf{X})$, $\nu = (\nu_1, \ldots, \nu_n)$.

THEOREM 2. *For $i \neq j$,*

$$0 \leq \mathrm{Cov}(S_i, S_j) \leq ((n-1)/n)\,\mathrm{Var}\, S_1 \tag{2.1}$$

with equality above iff S_1 is linear in functions of its individual arguments.

Proof. Without loss of generality, assume that the S_i (which are identically distributed) have zero mean. Accordingly, we consider ES_1S_{n+1} as a surrogate for $\mathrm{Cov}(S_i, S_j)$, $i \neq j$. Using the basis given above and symmetry considerations yields

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