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PROBABILITY MEASURES ON THE CIRCLE AND THE ISOPERIMETRIC INEQUALITY

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The theory of planar convex sets invokes measures of a certain class. Accordingly, the isoperimetric inequality can be translated into quadratic inequalities for probability measures on the unit circle.

Various tools from convex geometry have been put to highly effective use in probability theory. One example is in the application by Anderson (1955) of the Brunn-Minkowski inequality to the probability content of symmetric, convex sets (see Tong (1980, chapter 4) for this and related results). A second instance is the resolution by Egorychev and Falikman of the van der Waerden conjecture by means of mixed discriminants, which arose originally in the study of mixed volumes (see Lagarias (1982)). The latter have recently been applied to combinatorial questions (Stanley (1981)).

That convex geometry and probability should be linked is not too surprising since each has a strong concern with the notion of positivity. What geometers have appreciated for some time, and what perhaps awaits systematic exploitation by probabilists, is that this link can be made concrete. This goes by the historical name of Minkowski's problem (Busemann (1958, pp. 60–67)). Roughly speaking, each compact, convex set in $\mathcal{R}^n$ can be identified with a bounded, positive measure on the unit sphere in that space. From the probabilist's point of view, a wide class of probability measures on the unit sphere *can be realized as compact, convex sets*. Existence of atoms, modes of convergence, and even statistical procedures have natural geometric analogs.

The author will treat some of these questions elsewhere. Here the flavor of the connection will be given by deriving two inequalities for probability measures μ on the unit circle $C = [0,2\pi)$.

INEQUALITY I.

(I) $$\int_C \int_C g(\theta-\lambda)\mu(d\theta)\mu(d\lambda) \leq (2\pi)^{-1}$$

where $g(\lambda) = [2(\pi-\lambda)\sin\lambda - \cos\lambda]/4\pi$ on $[0,2\pi)$ and is extended 2π periodically. Equality holds iff μ admits the representation

$$\mu(d\lambda) = [(2\pi)^{-1} + c_1\cos\lambda + c_2\sin\lambda]d\lambda$$

for constants c_1, c_2.

INEQUALITY II.

(II) $$\int_C \int_C |\sin(\theta-\lambda)|\mu(d\theta)\mu(d\lambda) \leq 2/\pi$$

with equality iff $1/2[\mu(d\lambda) + \mu(d(\lambda+\pi))] = (2\pi)^{-1}\, d\lambda$.

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