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SOME SHARP MARTINGALE INEQUALITIES RELATED TO DOOB'S INEQUALITY

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Let $p > 1$. The best constant $C = C_{n,p}$ in the inequality $E(\max_{1 \le i \le n} |Y_i|)^p \le C\, E|Y_n|^p$, where $Y_1, \ldots, Y_n$ is a martingale, is determined. For each n and p, the method allows one to construct a martingale attaining equality. As $n \to \infty$, $K_p n^{2/3}(q^p - C_{n,p}) \to 1$, where K_p is a known constant. As an application, the classical inequality of Doob is sharp. It is shown that equality cannot be attained by a non-zero martingale.

1. Introduction. Let $Y_1, Y_2, \ldots$ be a martingale with difference sequence $X_1 = Y_1$, $X_i = Y_i - Y_{i-1}$, $i = 2, 3, \ldots$ Thus, $E(X_i|X_1, \ldots, X_{i-1}) = 0$, $i = 2, 3, \ldots$. Let $p > 1$ and define $q = p/(p-1)$. The principal purpose of this paper is to determine the best constant $C = C_{n,p}$ in the inequality

$$E(\max_{1 \le i \le n} |Y_i|)^p \le CE|Y_n|^p. \tag{1.1}$$

Although $C_{n,p}$ is found in implicit form, it can be easily approximated. For each n and p, the method allows one to construct a martingale attaining equality in (1.1), with $C = C_{n,p}$. Once the distribution of Y_1 is fixed, such a martingale is uniquely determined.

Furthermore, as $n \to \infty$, $C_{n,p} \to q^p$ at a rate proportional to $n^{-2/3}$. Specifically, $K_p n^{2/3}$ $(q^p - C_{n,p}) \to 1$, where K_p is a known constant. As an application, this provides a new proof that Doob's inequality (1953, p. 317)

$$E(\sup_{i \ge 1} |Y_i|)^p \le q^p \sup_{i \ge 1} E|Y_i|^p \tag{1.2}$$

is sharp. An example to that effect was given previously by Dubins and Gilat (1978). Inequality (1.2) is strengthened to

$$E(\sup_{i \ge 1} |Y_i|)^p \le q^p \sup_{i \ge 1} E|Y_i|^p - q\, E|Y_1|^p. \tag{1.3}$$

It follows from (1.3) that equality cannot be attained in (1.2) by a non-zero martingale. The sharpness of Doob's inequality for $p = 1$ (1953, p. 317)

$$E(\sup_{i \ge 1} |Y_i|) \le [e/(e-1)](1 + E(\sup_{i \ge 1} |Y_i| \log^+ \sup_{i \ge 1} |Y_i|)],$$

is still an open question.

The method of this paper is based on results from the theory of moments (Kemperman (1968)), together with induction and the device of conditioning. Where applicable, it always leads to a sharp inequality and provides an example of a martingale attaining equality or nearly so. In principle, the method can be applied to many other martingale inequalities. For example, the author used it (Cox (1982)) to find the best constant in Burkholder's weak-L^1 inequality (Burkholder (1979)) for the martingale square function. The method does have the drawback of computational complexity, which sometimes makes it difficult or impossible to push the calculations through.

Section 2 contains statements of the results, together with comments and some proofs. In section 3, some needed analytic lemmas are established. Section 4 contains the main proofs, and an example for the case $p = 2$, $n = 3$ of (1.1).

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