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A CLASS OF GENERALIZATIONS OF HÖLDER'S INEQUALITY

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Let $a_1 \geqslant a_2 \geqslant \ldots \geqslant a_n \geqslant 0$, $b_1 \geqslant b_2 \geqslant \ldots \geqslant b_n \geqslant 0$ and consider the problem of maximizing $\Sigma_1^n a_i b_i$ subject to $\Sigma_1^m a_i^p = 1$, $m \leqslant n$. In this paper Kuhn-Tucker theory is used to solve the problem and consequently to obtain a generalization of Hölder's inequality. The reversal of the generalized inequality, its extension to the symmetric gauge functions and the continuous case are discussed. Some statistical applications and other work presently in progress are outlined.

1. Introduction and Summary. In an article published in 1889, O. Hölder presented two basic and now very well known results. The first of these is known as "Jensen's Inequality". In an addendum to his article J. L. W. V. Jensen (1906), who is credited with its discovery, acknowledges that the inequality is not "entirely new", that, after completing his work, through a monograph by A. Pringsheim he became aware of its earlier discovery by Hölder (1889). In the same 1906 paper, Jensen uses this Hölder-Jensen inequality for convex functions to derive in explicit form the second basic result only implicit in Hölder (1889), namely the "Hölder's inequality" bounding the inner products of vectors in terms of their norms. Specifically, if **a** and **b** are two vectors with nonnegative components a_1, $a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ respectively, then Hölder's inequality asserts that

$$\Sigma_{i=1}^n a_i b_i \leqslant (\Sigma_1^n a_i^p)^{1/p} (\Sigma_1^n b_i^q)^{1/q}, \tag{1.1}$$

for any $p \geqslant 1$ and q satisfying $p^{-1} + q^{-1} = 1$. The inequality is reversed if $p < 1$, provided the components of **a** and **b** are strictly positive. Moreover, if these components are proportional, i.e. $a_i^p = cb_i^q$ for some c and $i = 1, 2, \ldots, n$ then in (1.1) and its reversal the equality holds. In this essay our interest centers on this classical inequality due to Hölder. Our objective is to present some recent generalizations of this inequality, to outline some statistical applications and to indicate the directions of further work which is in progess.

Although Hölder's inequality (1.1) was introduced as a theorem about the "mean values" it is now widely studied in its own right and is variously applied. In its better known applications in sciences, it is generally encountered as the particular case $p = 2$, i.e. the Cauchy-Schwarz inequality. In mathematics it appears in the theory of linear spaces in the context of identifying the conjugate or adjoint spaces and establishing their dual character. For discussions of various generalizations of (1.1) see Beckenbach and Bellman (1965), Hardy, Littlewood, and Polya (1952), Mitronović (1968) and Rockafellar (1970). The generalizations include sharp bounds on the sums of products of type $\Sigma_{i=1}^n a_i b_i c_i$ of the components of three or more vectors, and on integrals of type $\int a(x)\, b(x) dx$. Another approach to generalizing (1.1) is to use arbitrary norms $\phi(\mathbf{a}) = \phi(a_1, a_2, \ldots, a_n)$ leading to results of the type

$$\Sigma_{i=1}^n a_i b_i \leqslant \phi(\mathbf{a})\, \phi^\circ(b), \tag{1.2}$$

where

$$\phi^\circ(\mathbf{b}) = \max_{a \neq 0} \Sigma_{i=1}^n a_i b_i / \phi(\mathbf{a}), \tag{1.3}$$

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