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ON TP_2 AND LOG-CONCAVITY

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Inter-relations between the TP_2 property and log-concavity of density functions have been investigated. The general results are then applied to noncentral chi-square density functions and beta density functions.

1. Results on Density Functions

Definition 1. A function f: $\mathcal{R}^2 \to \mathcal{R}$ is said to be TP_2 (Karlin (1968)) if, for $x_1 < x_2$, $y_1 < y_2$

$$f(x_1,y_2)f(x_2,y_1) \leq f(x_1,y_1)f(x_2,y_2). \tag{1.1}$$

We shall say that $1/f$ is TP_2, if (1.1) holds for f with the inequality reversed.

Let X be a positive random variable having the p.d.f. $f(\cdot,\theta,\lambda)$ with respect to Lebesgue measure; $\theta > 0$, $\lambda \geq 0$.

Definition 2. The p.d.f. $f(x,\theta,\lambda)$ is said to have the reproductive property (RP) in θ, if there exists a distribution function $G(\cdot,s)$ on $\mathcal{R}^+$ $(s > 0)$ such that

$$\int_0^x f(x-y,\theta,\lambda)\, G(dy,s) = f(x,\theta+s,\lambda). \tag{1.2}$$

THEOREM 1. *Suppose $f(x,\theta,\lambda)$ has the RP in θ. Then* (i) $f(x,\theta,\lambda)$ TP_2 *in* $(x,\lambda) \to 1/f(x,\theta,\lambda)$ TP_2 *in* (θ,λ), (ii) $f(x,\theta,\lambda)$ TP_2 *in* $(x,\theta) \to f(x,\theta,\lambda)$ *log-concave in* θ.

Proof. (i) For $0 < x_1 < x_2$, $\lambda_1 < \lambda_2$ we have

$$f(x_2,\theta,\lambda_1)f(x_1,\theta,\lambda_2) \leq f(x_2,\theta,\lambda_2)f(x_1,\theta,\lambda_1). \tag{1.3}$$

Write $x_1 = x_2 - y$. Integrating (1.3) with respect to $G(dy,s)$ we get

$$f(x_2,\theta,\lambda_1)f(x_2,\theta+s,\lambda_2) \leq f(x_2,\theta,\lambda_2)f(x_2,\theta+s,\lambda_1), \tag{1.4}$$

which shows that $1/f(x,\theta,\lambda)$ is TP_2 in (θ,λ).

(ii) For $0 < x_1 < x_2$, $\theta_1 < \theta_2$, we have

$$f(x_1,\theta_2,\lambda)f(x_2,\theta_1,\lambda) \leq f(x_2,\theta_2,\lambda)f(x_1,\theta_1,\lambda). \tag{1.5}$$

Write $x_1 = x_2 - y$. Integrating (1.5) with respect to $G(dy,s)$ we get

$$f(x_2,\theta_2+s,\lambda)f(x_2,\theta_1,\lambda) \leq f(x_2,\theta_2,\lambda)f(x_2,\theta_1+s,\lambda), \tag{1.6}$$

which shows that $f(x,\theta,\lambda)$ is log-concave in θ. □

Definition 3. The p.d.f. $f(x,\theta,\lambda)$ is said to have the mixture property (MP) in (θ,λ) if there exists a non-negative random variable K with the distribution $H(\cdot,\tau)$ with $\tau > 0$ such that

$$\int_0^\infty f(x,\theta+k,\lambda)\, H(dk,\tau) = f(x,\theta,\lambda+\tau). \tag{1.7}$$

Suppose H in Definition 3 possesses a density function h with respect to a σ-finite measure ν.

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