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RANDOM REPLACEMENT SCHEMES AND MULTIVARIATE MAJORIZATION[1]

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In this note we obtain certain inequalities comparing random replacement schemes to sampling with replacement. Some of the results are related to multivariate majorization and Schur functions.

1. Various Stochastic Comparisons and Random Replacement Schemes. Let $\mathscr{A} = \{a_1, \ldots, a_N\}$, $a_i \in \mathscr{R}$ = the real line. We shall consider a sample of size n ($n \leq N$) drawn from A, and denote the observations by $X_1, \ldots, X_n$. In a symmetric random replacement scheme the observation X_1 is drawn with equal probabilities from A, i.e., $P(X_1 = a_i) = 1/N$, $i = 1, \ldots, N$. The element drawn for X_1 is replaced in A with probability π_1, and removed from A with probability $1-\pi_1$. Then X_2 is sampled, and the element which is drawn is replaced with probability π_2. Continuing to X_{n-1}, the vector $\pi = (\pi_1, \ldots, \pi_{n-1})$ defines the random replacement scheme $R(\pi)$. Note that for $\pi = \mathbf{0} = (0, \ldots, 0)$, $R(\pi)$ is equivalent to sampling without replacement while for $\pi = \mathbf{1} = (1, \ldots, 1)$, $R(\pi)$ corresponds to sampling with replacement and $X_1, \ldots, X_n$ are i.i.d.

It follows from Joag-Dev and Proschan (1983) that under $R(\mathbf{0})$, $X_1, \ldots, X_n$ are *negatively associated, i.e.,*

$$E\{\phi(X_i, i \in A)\psi(X_j, j \in B)\} \leq E\phi(X_i, i \in A)E\psi(X_j, j \in B) \tag{1.1}$$

for any partition A,B of $1, \ldots, n$, where ϕ and ψ are increasing functions.

In particular, (1.1) implies

$$E\{\Pi_{i=1}^n \varphi_i(X_i)\} \leq \Pi_{i=1}^n E\varphi_i(X_i) \tag{1.2a}$$

for any functions φ_i, all increasing (or all decreasing) and nonnegative. Note that (1.2a) can be written as

$$E_{R(\mathbf{0})}\{\Pi_{i=1}^n \varphi_i(X_i)\} \leq E_{R(\mathbf{1})}\{\Pi_{i=1}^n \varphi_i(X_i)\}. \tag{1.2b}$$

Inequalities for sampling schemes were obtained by various authors including Sen (1970), Rosén (1972), Serfling (1973), Kemperman (1973), Karlin (1974), Van Zwet (1983), and Krafft and Schaefer (preprint). The question of characterizing the class of functions for which

$$E_{R(\pi)}\psi(X_1, \ldots, X_n) \leq E_{R(\mathbf{1})}\psi(X_1, \ldots . X_n) \tag{1.3}$$

remains unresolved. The next result provides a class of functions for which (1.3) holds.

THEOREM 1. $\mathbf{E}_{R(\pi)}\{\Pi_{i=1}^n \varphi(X_i)\} \leq \mathbf{E}_{\mathbf{R(1)}}\{\Pi_{i=1}^n \varphi(X_i)\}$ *for any* $\varphi \geq 0$.

Proof. We write π instead of $R(\pi)$ as an index for the expectation. For $n = 2$

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