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INVARIANT ORDERING AND ORDER PRESERVATION[1]

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Suppose $\mathcal{G}$ is a group of one-to-one transformations of a set $\mathcal{X}$ onto $\mathcal{X}$, M is maximal invariant taking values in an ordered set $(\mathcal{M}, \succeq_M)$, and $\succeq$ is an ordering on $\mathcal{X}$ induced from $\succeq_M$. Properties of $(\mathcal{X}, \succeq)$ are studied in Part I including lattice properties and order preservation. Examples include an ordering on $\mathcal{F}_{n\times m}$ having properties of Loewner's (1934) ordering for Hermitian varieties, a unitary ordering on $\mathcal{F}_{n\times m}$ giving a lattice, and orderings on $\mathcal{R}^n$ invariant under various groups. Applications to a variety of problems in statistics and applied probability are given in Part II..

PART I. THEORY

1. Introduction. Many problems exhibit symmetries as invariance under a group $\mathcal{G}$ acting on a set $\mathcal{X}$. Invariance principles require that solutions be invariant, and reduction by invariance preserves essentials while discarding irrelevant details. Because order relations often assume a prominent role in the analysis of such problems, it is instructive to consider orderings symmetric under $\mathcal{G}$.

Often $\mathcal{X}$ is finite-dimensional; examples are the Euclidean space $\mathcal{R}^n$, the linear space $\mathcal{F}_{n\times m}$ of $(n\times m)$ matrices over the complex field $\mathcal{C}$, the Hermitian $(n\times n)$ matrices $\mathcal{H}_n$, and the cone $\mathcal{H}_n^+$ of positive semidefinite Hermitian varieties. Typical groups of transformations are the classical groups. An ordering on $\mathcal{H}_n$ in wide usage was studied by Loewner (1934) in which $\mathbf{A} \succeq_L \mathbf{B}$ on $\mathcal{H}_n$ if and only if $\mathbf{A} - \mathbf{B} \in \mathcal{H}_n^+$. This ordering is invariant under the general linear group $Gl(n)$ acting on $\mathcal{H}_n$ by congruence, for $\mathbf{A} \succeq_L \mathbf{B}$ on $\mathcal{H}_n$ if and only if $\mathbf{CAC}^* \succeq_L \mathbf{CBC}^*$ on $\mathcal{H}_n$ for every $\mathbf{C} \in Gl(n)$, with $\mathbf{C}^*$ the conjugate transpose of $\mathbf{C}$. The relation $\succeq_L$ as an ordering on $\mathcal{F}_{n\times n}$ was considered by Hartwig (1976).

Here we study symmetric orderings induced through maximal invariants, the preservation of such orderings, and the possible transitivity of lattice properties. Our principal motivation stems from needed orderings on all of $\mathcal{F}_{n\times m}$ and not just $\mathcal{F}_{n\times n}$ or its Hermitian varieties.

2. The Basic Results. A set $\mathcal{X}$ together with a binary relation $\succeq$ is said to be *linearly ordered* if the relation is reflexive, transitive, antisymmetric, and complete. The relation is a *partial ordering* if it is reflexive, transitive, and antisymmetric, and a *preordering* if it is reflexive and transitive. A partially ordered set $(\mathcal{X}, \succeq)$ is a *lower semi-lattice* if for any two elements x,y there is an element $v = x \wedge y \in \mathcal{X}$ that is a greatest lower bound for x,y; an *upper semi-lattice* if there is a least upper bound $u = x \vee y$ for x,y in $\mathcal{X}$; and a *lattice* if it is both a lower and upper semi-lattice.

Let $\mathcal{G}$ be a group of one-to-one transformations from $\mathcal{X}$ onto $\mathcal{X}$. A function f on $\mathcal{X}$ is said to be *invariant* under $\mathcal{G}$ if, for any $(x,g) \in \mathcal{X}\times\mathcal{G}$, $f(gx) = f(x)$, and to be *maximal invariant* if it is invariant and if $f(x) = f(y)$ implies $y = gx$ for some $g \in \mathcal{G}$. The $\mathcal{G}$-*orbit* of $x_0 \in \mathcal{X}$

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