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ON GROUP INDUCED ORDERINGS, MONOTONE FUNCTIONS, AND CONVOLUTION THEOREMS[1]

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Orderings defined by compact groups of linear transformations acting on vector spaces are studied. In some cases, these orderings induce orderings on convex cones similar to those defined by reflection groups. In these cases the monotone functions can be conveniently characterized. Convolution theorems for monotone functions are discussed.

1. Introduction. Majorization, as defined by Hardy, Littlewood and Pólya (1934), has been an extremely important notion in the theory and applications of many types of inequalities. The recent work of Marshall and Olkin (1979) contains an extensive discussion of majorization and its application to many branches of mathematics including probability and statistics. Although not essential for understanding this paper, the reader may find it useful to glance through Part I of Marshall and Olkin (1979).

To motivate the situation to be considered here, first recall the permutation group definition of majorization (see Rado (1952)). Let $\mathcal{P}_n$ be the group of $n \times n$ permutation matrices acting on $\mathcal{R}^n$. For $x,y \in \mathcal{R}^n$, x is majorized by y (written as $x \leq y$) means that x is in the convex hull of the set $\{gy|g\in\mathcal{P}_n\}$ (the $\mathcal{P}_n$-orbit of y). A careful study of the pre-order $\leq$ (using the terminology in Marshall and Olkin (1979), p. 13) has resulted in a useful and important characterization of the real valued functions f which are decreasing or increasing in the pre-order of majorization (see Schur (1923), Ostrowski (1952)). A recent result of Marshall and Olkin (1974), which has had applications in probability and statistics, shows that the convolution of two decreasing (in the pre-order of majorization) functions is again a decreasing function.

In this paper, we begin a systematic study of pre-orderings defined on vector spaces which arise in much the same way that majorization arises. Let G be any closed group of $n \times n$ orthogonal matrices. Using G, rather than $\mathcal{P}_n$, define a pre-order on $\mathcal{R}^n$ as follows: $x \leq y$ iff x is in the convex hull of $\{gy|g\in G\}$. The examples in the next section show that there are a number of groups G which give useful and interesting orderings. Based on the known majorization results, it seems rather natural to ask for conditions on G for which (i) it is possible to characterize the class of decreasing real valued functions on $\mathcal{R}^n$. (ii) the convolution result of Marshall and Olkin (1974) continues to hold.

This paper is mainly concerned with (i), but (ii) is discussed rather incompletely. Here is a brief outline of the paper. In Section 2, group induced orderings are defined on inner product spaces. The geometry which prevails in the permutation group case is described and is shown to hold in a number of interesting cases. It is this geometry which is used in Section 3 to give a characterization of the decreasing functions. The results of Marshall, Walkup and Wets (1967) on cone orderings are used extensively in Section 3. In Section

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