4. Functional Central Limit Theorems.

In the last section we have already mentioned Donsker's functional central limit theorem for the uniform empirical process $\alpha_n \equiv (\alpha_n(t))_{t\in[0,1]}$, where $\alpha_n(t) = n^{1/2}(U_n(t) - t)$, $U_n(t)$ being the empirical df based on independent random variables η_i having uniform distribution on the sample space $X = [0,1]$ with its Borel σ-algebra $\mathcal{B} = [0,1] \cap \mathbb{B}$.

In the setting of an empirical $\mathcal{C}$-process $\beta_n \equiv (\beta_n(C))_{C\in\mathcal{C}}$ the uniform empirical process α_n is a very special case taking $\mathcal{C} = \{[0,t]: t \in [0,1]\}$ and identifying $\alpha_n(t)$ with $\beta_n(C) = n^{1/2}(\mu_n(C) - \mu(C))$ for $C = [0,t]$, μ_n being the empirical measure based on $\eta_1,\dots,\eta_n$ and μ being the uniform distribution on $[0,1]$; note that $\mu_n(C) = U_n(t)$ and $\mu(C) = t$ for $C = [0,t]$.

The present section is concerned with some extensions of Donsker's functional central limit theorem in its form (44)(ii) to more general situations.

FUNCTIONAL CENTRAL LIMIT THEOREMS FOR EMPIRICAL $\mathcal{C}$-PROCESSES:

Let $X = (X,\mathcal{B})$ be an arbitrary measurable space considered as a sample space for a given sequence $\xi_1,\xi_2,\dots$ of i.i.d. random elements in $(X,\mathcal{B})$, the ξ_i's being defined on some common p-space $(\Omega,\mathcal{F},\mathbb{P})$ with law μ on $\mathcal{B}$. If not stated otherwise we will consider the canonical model

$$(\Omega,\mathcal{F},\mathbb{P}) = (X^{\mathbb{N}}, \mathcal{B}_{\mathbb{N}}, \times_{\mathbb{N}} \mu)$$

with the ξ_i's being the coordinate projections of $X^{\mathbb{N}}$ onto X.

Let $\mu_n(B) = \frac{1}{n} \sum_{i=1}^{n} 1_B(\xi_i)$, $B \in \mathcal{B}$, be the empirical measure based on $\xi_1,\dots,\xi_n$.